**Aim lecture:** We define the determinant via permutations.

**Fact**

Let $f(x_1, \ldots, x_n)$ be an $\mathbb{R}$-valued function & $\sigma \in S_n$. Then

$$\sigma . (-f) = - (\sigma . f).$$

**Proof.**
We begin with the difference product

$$\Delta_n(x_1, \ldots, x_n) = \prod_{i<j}(x_i - x_j)$$

e.g. $\Delta_2(x_1, x_2) = x_1 - x_2, \Delta_3$

**Thm-Defn**

Let $\sigma \in S_n$ be a product of $m$ 2-cycles. Then $\sigma \Delta_n = (-1)^m \Delta_n$.

- We define the *sign* of $\sigma$ to be $\text{sgn}(\sigma) = (-1)^m$ which is well-defined.
- We say $\sigma$ is *even* if $\text{sgn}(\sigma)$ is 1 (so $\sigma$ is a product of an even number of 2-cycles) and *odd* otherwise.

E.g. $(1 \ 2) \Delta_3$
It suffices to show that for any 2-cycle \((i j)\) we have \((i j) \cdot \Delta_n = -\Delta_n\) for repeating this \(m\) times & using fact page 1, gives the desired result.

We can assume \(i < j\). We observe the effect of \((i j)\) on each factor of \(\Delta_n\):

- \((i j) \cdot (x_i - x_j) = -(x_i - x_j)\) factor negated
- For \(r, s, i, j\) disjoint, \((i j) (x_r - x_s) = x_r - x_s\) no change
- For \(r < i < j\), \((i j) (x_r - x_i)(x_r - x_j) = (x_r - x_j)(x_r - x_i)\) no change
- Sim no change for \((x_i - x_r)(x_r - x_j)\) when \(i < r < j\) and for \((x_i - x_r)(x_j - x_r)\) when \(i < j < r\).

Multiplying the above gives the thm.

**Corollary**

\[
\text{sgn}(\sigma \tau) = \text{sgn}(\sigma)\text{sgn}(\tau)
\]

**Proof** If \(\sigma, \tau\) are products of \(m, m'\) 2-cycles resp then, \(\sigma \tau\) is a product of \(m + m'\) 2-cycles. Cor boils down to \((-1)^{m+m'} = (-1)^m(-1)^{m'}\).
Permuting terms in sums and products

Recall $J_n = \{1, \ldots, n\}$. For $\sigma \in S_n$, we may re-arrange terms to obtain formulas like

$$\sum_{i \in J_n} a_i = \sum_{i \in J_n} a_{\sigma(i)}$$

$$\prod_{i \in J_n} a_i = \prod_{i \in J_n} a_{\sigma(i)}$$

We’ll also need following

**Lemma**

Let $\tau \in S_n$. Then as $\sigma$ runs through the elts of $S_n$ (once)

1. $\tau \sigma$ runs through all the elts of $S_n$ exactly once.
2. $\sigma^{-1}$ runs through all the elts of $S_n$ exactly once.

**Proof.** (1) follows from prop lect 3, on the regular action of $S_n$. (2) is easy ex.
Determinant

Defn

For $A = (a_{ij}) \in M_{nn}(\mathbb{F})$ we define the determinant of $A$ to be

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i \in J_n} a_{i, \sigma(i)}$$

E.g. 1 $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =$

E.g. 2 Suppose $A = (a_{ij})$ is upper triangular i.e. $a_{ij} = 0$ if $j < i$. 
$\det(A) =$
Position of entries in term

Consider a monomial term $\prod_{i \in J_n} a_{i \sigma(i)}$ of the determinant of $(a_{ij})$.

Note that since $i$ ranges over $J_n$, there’s exactly one entry $a_{i \sigma(i)}$ from each row.

Since $\sigma(i)$ ranges over elements of $J_n$ exactly once, there’s exactly one entry from each column.

E.g.
Alternating & multi-linear functions

The new defn of determinant agrees with the one from first year. Need some properties to see this.

**Defn**

Let \( f : M_{nn}(\mathbb{F}) \longrightarrow \mathbb{F} \) be a function.

1. We say \( f \) is **multi-linear** in the columns if for any two matrices of form \( A_1 = (A \ v_1 \ A') \), \( A_2 = (A \ v_2 \ A') \in M_{nn}(\mathbb{F}) \) with \( i \)-th columns \( v_1, v_2 \) resp (\& other columns the same) we have
   \[
   f(A(v_1 + v_2)A') = f(A_1) + f(A_2) \quad \& \quad f(A(cv_1)A') = cf(A_1) \text{ for } c \in \mathbb{F}.
   \]
   Here \( A \) represents first \( i - 1 \) columns of \( A_1, A_2 \) whilst \( A' \) represents last \( n - i \).

2. We say \( f \) is **alternating** in the columns if for any 2-cycle \( \tau \) we have
   \[
   f(a_{i\tau(j)})_{ij} = -f(a_{ij}) \text{ i.e. swapping two columns of a matrix negates the value of } f \quad \& \text{ furthermore in char 2, } f(A) = 0 \text{ if two columns are equal).}
   \]

3. There are similar defns for rows.

**e.g.** \( \det : M_{22} \longrightarrow \mathbb{F} \).
Key properties of the determinant function

**Theorem**

*The determinant function is multi-linear and alternating in the rows and columns.*

Proof is mainly exercise. We check alternating in columns. For 2-cycle $\tau$ note that $\tau^{-1} = \tau$ has sign $-1$.

\[
\begin{align*}
\det(a_{i\tau(j)}) &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i \in J_n} a_{i\tau \sigma(i)} \\
&= \sum_{\rho \in S_n} \text{sgn}(\tau^{-1} \rho) \prod_{i \in J_n} a_{i\rho(i)} \\
&= \text{sgn}(\tau^{-1}) \sum_{\rho \in S_n} \text{sgn}(\rho) \prod_{i \in J_n} a_{i\rho(i)} \\
&= -\det(a_{ij})
\end{align*}
\]

Note the use of the lemma in line 2 (where we changed var to $\rho = \tau \sigma$) and corollary in line 4.
**Defn**

Let \( A = (a_{ij}) \in M_{nn} \) and \( i, j \in J_n \).

1. The \((i, j)\)-th minor of \( A \) is the matrix \( A(i,j) \in M_{n-1,n-1} \) obtained by deleting the \( i \)-th row & \( j \)-th columns from \( A \).

2. The \((i, j)\)-th cofactor of \( A \) is \((-1)^{i+j} \det A(i,j)\).

**e.g.**
Connection with Laplace expansions

The equivalence with the defn via Laplace expansions is given in

**Thm**

Let $A = (a_{ij})$ and $C_{ij}$ denote its $(i,j)$-th cofactor. Then for any fixed $i$

$$
\det A = \sum_{j=1}^{n} a_{ij} C_{ij}.
$$

The proof (omitted) is easily obtained by multi-linearity and best seen in an actual example.