**Aim lecture:** We look at some applications of singular value decomposition.

**Q** Let $Q \in M_{nn}(\mathbb{C})$ be a matrix that we know from theory to be unitary. Suppose we have some experiment to determine its entries, and the resulting matrix is $Q_0$. With measurement error, it is unlikely that $Q_0$ is unitary. Can we guess $Q$ from $Q_0$?

**A** Let $Q_0 = U_lDU_r^*$ be a singular value decomposition (SVD) of $Q_0$. Then $Q \approx U_lDU_r^* \implies D \approx U_l^*QU_r$. Thus $D$ is diagonal with non-negative entries & is approximately unitary so must be close to $I_n$. This suggests

**Fact**

The best unitary matrix approximating $Q_0$ is $U_lU_r^*$.
Least squares in the diagonal case

Suppose $D$ is a real $m \times n$ diagonal matrix of the form $(d_{ij}) = D = \begin{pmatrix} D_+ & 0 \\ 0 & 0 \end{pmatrix}$, where $D_+$ is invertible $\rho \times \rho$ and the zero matrices have size $\rho \times (n - \rho), (m - \rho) \times \rho, (m - \rho) \times (n - \rho)$. Hence $D_+$ has non-zero diagonal entries $d_{11}, \ldots, d_{\rho\rho}$.

Prop

A least squares solution to the eqn $Dv = w$ is given by

1. any $v$ with co-ord $v_1 = d_{11}^{-1} w_1, \ldots, v_\rho = d_{\rho\rho}^{-1} w_\rho$ and $v_{\rho+1}, \ldots, v_n$ arbitrary.
2. The solution $v$ with minimal length $\|v\|$ is $v = (d_{11}^{-1} w_1, \ldots, d_{\rho\rho}^{-1} w_\rho, 0, \ldots, 0)^T$.

Proof. Defining $d_{ii} = v_i = 0$ if $i > n$, the least squares solns are those minimising

$$\|Dv - w\|^2 = \sum_{i=1}^{m} (d_{ii} v_i - w_i)^2 = \sum_{i=1}^{\rho} (d_{ii} v_i - w_i)^2 + \sum_{i=\rho+1}^{m} w_i^2.$$ 

This is minimised precisely when $d_{ii} v_i - w_i = 0$ for $i = 1, \ldots, \rho$. 

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Generalised inverses of diagonal matrices

Let $D$ be the diagonal matrix on the last slide and consider the $n \times m$-diagonal matrix

$$D^- = \begin{pmatrix} D_{++}^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

We can restate Prop 2) as

**Defn-Upshot**

The unique minimal length least squares soln to $Dv = w$ is $v = D^-w$. We call $D^-$ the (Moore-Penrose) generalised inverse of $D$.

Note that

$$DD^- =$$

**Q** How do you extend this to arbitrary matrices?

**A** Consider the linear map $T_A : \mathbb{R}^n \to \mathbb{R}^m$, and use an o/n change of co-ord on $\mathbb{R}^n, \mathbb{R}^m$ so that the representing matrix is diagonal, i.e. SVD.
Let $A \in M_{mn}(\mathbb{R})$ and $A = U_l D U_r^*$ be a SVD for $A$.

**Defn**

The (Moore-Penrose) generalised inverse of $A$ is $A^* = U_r D^{-1} U_l^*$. (It turns out it does not depend on the SVD!)

**E.g.** Find the gen inverse of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
Consider the SVD $A = U_l D U_r^\ast$ as above.

**Prop**
The least squares soln to $Av = w$ which has minimal length is $v = A^{-w}$.

**Proof.** We seek to minimise $\|Av - w\| = \|U_l D U_r^\ast v - w\| = \|D U_r^\ast v - U_l^\ast w\|$ since $U_l$ is orthogonal. We change variables to $x = U_r^\ast v$ and seek to minimise $\|Dx - U_l^\ast w\|$. Since $U_r$ is orthogonal, the minimal length solns $x$ correspond to the minimal length solns $v$. This is given by

$$x = D^{-1} U_l^\ast w.$$ 

Hence the minimal length least squares soln is

$$v = U_r D^{-1} U_l^\ast w = A^{-1} w.$$