**Aim lecture:** We use the spectral thm for normal operators to show how any orthogonal matrix can be built up from rotations & reflections.

In this lecture we work over the fields $\mathbb{F} = \mathbb{R} & \mathbb{C}$. We let

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

the $2 \times 2$-matrix which rotates anti-clockwise about $(0,0)$ through angle $\theta$. Note that $\det R_\theta = 1$ & that $R_\pi$ is the diagonal matrix $-I_2 = (-1) \oplus (-1)$.

**Defn**

Let $A \in M_{nn}(\mathbb{R})$. We say that $A$ is a *rotation* matrix if it is orthogonally similar to $I_{n-2} \oplus R_\theta$ for some $\theta$. We say that $A$ is a *reflection* matrix if it is orthogonally similar to $I_{n-1} \oplus (-1)$.

**Rem** Note that this generalises the usual notions in dim 2 & 3. Also, rotations are orthogonal with determinant 1 whilst reflections are orthogonal with $\det -1$. 

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Lecture 42: Orthogonal matrices & rotations  
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Recall that orthogonal matrices have det ±1. We defer the proof of

**Theorem**

1. Let $A \in O_2$. If $\det A = 1$ then $A$ is orthog sim to $R_\theta$ for some $\theta$ (so $\text{tr } A = 2 \cos \theta$) & if $\det A = -1$ then $A$ is a reflection.

2. Let $A \in O_3$. If $\det A = 1$ then $A$ is orthog sim to $(1) \oplus R_\theta$ (so $\text{tr } A = 1 + 2 \cos \theta$) & if $\det A = -1$ then $A$ is orthog sim to $(-1) \oplus R_\theta$ (so $\text{tr } A = -1 + 2 \cos \theta$) for some $\theta$.

**Corollary**

Let $A, B \in M_{22}(\mathbb{R})$ or $A, B \in M_{33}(\mathbb{R})$. If $A, B$ are rotation matrices, then so is $AB$. If $A, B$ are reflection matrices, then $AB$ is a rotn.

**Proof.** In both cases, $AB$ is an orthog matrix with det $(\pm 1)^2 = 1$ so the thm gives the result.
Example

E.g. Verify that the following is a rotation matrix & determine the axis of rotn & angle of rotn.

\[
A = \frac{1}{3} \begin{pmatrix}
1 & 2 & -2 \\
2 & 1 & 2 \\
2 & -2 & -1
\end{pmatrix}
\]
E-values & e-vectors of real matrices

Recall the conjugation map $(\cdot)^* : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a conjugate linear isomorphism. Furthermore, for $v, v' \in \mathbb{C}^n$ we have $\|v\| = \|v\| \& v \perp v' \iff \overline{v} \perp \overline{v'}$.

**Prop 1**

Let $A \in M_{nn}(\mathbb{R}) \& \lambda \in \mathbb{C}$.

1. The conjugation map restricts to a conjugate linear isomorphism $(\cdot)^* : E_\lambda \rightarrow E_{\overline{\lambda}}$, that is, $E_\lambda = E_{\overline{\lambda}}$. In particular, $\lambda$ is an e-value of $A$ iff $\overline{\lambda}$ is.

2. The conjugation map sends any basis for $E_\lambda$ to a basis for $E_{\overline{\lambda}}$.

**Proof.** 1) Let $v \in E_\lambda$. Then

$$A\overline{v} = \overline{A \, v} = \overline{A v} = \overline{\lambda v} = \overline{\lambda} \, \overline{v}.$$ 

Hence $E_\lambda \subseteq E_{\overline{\lambda}} \&$ the reverse inclusion follows from this result applied to $\overline{\lambda}$. Conjugation is injective so 1) follows.

2) follows from the fact that conjugate lin isom take bases to bases (just as is the case for lin isom). The proof is a mild modification of the lin case.
E-theory of $R_\theta$

To study rotations, we need to know the e-theory for $R_\theta$ well.

**Prop 2**

We may unitarily diagonalise (over $\mathbb{C}$) $R_\theta = U((e^{i\theta}) \oplus (e^{-i\theta}))U^*$ where

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}.$$ 

In particular, $E_{e^{i\theta}} = \mathbb{C} \left( \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}} \right)$, $E_{e^{-i\theta}} = \mathbb{C} \left( \frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}} \right) = \mathbb{C} \left( \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}} \right)$.

**Proof.** easy ex. Let’s just check the $e^{i\theta}$-e-space and use Prop 1.
General classification of orthogonal matrices

Theorem

Let \( A \in O_n \).

1. If \( \det A = 1 \) then \( A \) is orthogonally similar to \( I_r \oplus R_{\theta_1} \oplus \ldots \oplus R_{\theta_s} \) for some \( r \in \mathbb{N}, \theta_1, \ldots, \theta_s \in \mathbb{R} \).

2. If \( \det A = -1 \) then \( A \) is orthogonally similar to \((-1) \oplus I_r \oplus R_{\theta_1} \oplus \ldots \oplus R_{\theta_s} \) for some \( r \in \mathbb{N}, \theta_1, \ldots, \theta_s \in \mathbb{R} \).

Proof. This proof is a classic example of the use of complex numbers to answer questions about reals!

Note that \( A \) viewed as complex matrix is unitary & hence normal. Hence the e-values have modulus 1 & we may apply the spectral thm for normal operators to conclude there is an \( A \)-invariant orthogonal direct sum decomposition into e-spaces of the form

\[
\mathbb{C}^n = E_1 \oplus E_{-1} \oplus E_{e^{i\theta_1}} \oplus \overline{E_{e^{i\theta_1}}} \oplus \ldots \oplus E_{e^{i\theta_s}} \oplus \overline{E_{e^{i\theta_s}}}
\]

where we used the propn to re-write some of the e-spaces.
The theorem will follow if we can find an orthonormal basis of real vectors (abbrev to real orthonormal basis) such that wrt the corresp co-ord system \( P \in O_n \), the representing matrix \( P^T A P \) is a direct sum of rotation matrices & matrices of the form \( \pm I \). Indeed, the only thing left to do is replace all copies of \(-I_2\) with \( R_{\pi} \) to get the desired final form. The thm thus follows from the following more precise result.

**Theorem (Version 2 for purposes of proof)**

1. There are real orthonormal bases for \( E_1, E_{-1} \).
2. There is a real orthonormal basis for \( V_j = E_{e^{i\theta_j}} \oplus \overline{E_{e^{i\theta_j}}} \) such that the matrix representing \( A \) restricted to \( V_j \) wrt the corresp the co-ord system is \( R_{\theta_j} \oplus \ldots \oplus R_{\theta_j} \) (there are \( \dim E_{e^{i\theta_j}} \) copies of \( R_{\theta_j} \)).

**Proof.** 1) Note \( E_1 = \ker(A - I) \), the kernel of a matrix with real entries, so we may find an orthonormal basis for it consisting of real vectors. Sim \( E_{-1} \) has an orthonormal basis of real vectors. Part 1) follows.
2) To simplify notation, we drop the subscript $j$ so $\theta = \theta_j$, $V = V_j$ etc. Let 
$\{v_1, \ldots, v_m\}$ be an orthonormal basis for $E_{e^{i\theta}}$ so $\{v_1, \ldots, v_m, \overline{v_1}, \ldots, \overline{v_m}\}$ is an 
orthonormal basis for $V$. Hence $V$ is an orthogonal direct sum of the subspaces 
$W_i = \text{Span}(v_i, \overline{v_i})$ so it suffices to find a real orthonormal co-ordinate system for $W_i$. Using the o/n co-ordinate system $C = (v_i, \overline{v_i}) : \mathbb{C}^2 \rightarrow W_i$, the representing matrix is 

$$C^{-1} \circ T_A \circ C = (e^{i\theta}) \oplus (e^{-i\theta})$$

where $T_A : W_i \rightarrow W_i$ is the left multn by $A$ on $W_i$. Prop 2 then shows that for 

$$U = \begin{pmatrix}
1 \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{pmatrix} \quad \& \quad C_R = CU^*$$

we have 

$$C_R^{-1} \circ T_A \circ C_R = UC^{-1} \circ T_A \circ CU^* = U((e^{i\theta}) \oplus (e^{-i\theta}))U^* = R_\theta$$

Hence the thm follows from
Final lemma to complete proof

Lemma

The co-ord system \( C_\mathbb{R} \) is 1) real and, 2) orthonormal.

Note that \( C_\mathbb{R} = \left( \frac{v_l + v_l}{\sqrt{2}}, i \frac{v_l - v_l}{\sqrt{2}} \right) = (w_+ + w_-) \) say.

For 1), we just check \( w_\pm = w_\pm \).

For 2), just note that \( C_\mathbb{R} \) is a composite of isomorphisms of inner product spaces, so is an isomorphism of inner product spaces itself.