Semisimple operators

Aim lecture: We extend the spectral thm to the normal operators. This gives a characterisation of when complex matrices can be unitarily diagonalised.

In this lecture, $\mathbb{F}$ will be an alg closed field, & later $\mathbb{F} = \mathbb{C}$.

**Prop-Defn**

Let $T : V \rightarrow V$ be linear & $\dim V < \infty$. We say that $T$ is semisimple or a *semisimple operator* if $V$ is a direct sum of e-spaces of $T$, in other words, $T$ can be diagonalised.

1. The direct sum of semisimple operators is semisimple.
2. If $T$ is semisimple & $W \leq V$ is a $T$-invariant subspace then $T|_W : W \rightarrow W$ is also semisimple.

**Proof.** 1) is clear from the theory of diagn.

2) We argue by contradn. Let $T$ be semisimple so that all the gen e-spaces are equal to the corresponding e-spaces. Suppose that $T|_W$ is not semisimple & its $\lambda$-e-space $E_\lambda \leq W$ is not equal to the gen $\lambda$-e-space $E_\lambda(\infty) \leq W$. Then we can find $w \in E_\lambda(\infty) - E_\lambda$. But then the gen e-space $\lambda$-e-space of $T$ contains $w$ so by semisimplicity of $T$, $w$ is also a $\lambda$-e-vector. This contradn proves the propn.
E-spaces of commuting operators

For this result we do not need $\mathbb{F}$ to be alg closed.

**Prop**

Let $X, Y : V \rightarrow V$ be linear maps. Let $E^X_\lambda, E^Y_\mu$ denote e-spaces of $X, Y$ resp. If $X, Y$ commute, that is $X \circ Y = Y \circ X$, then $E^X_\lambda$ is $Y$-invariant & $E^Y_\mu$ is $X$-invariant.

**Rem** Hence the e-spaces are both $X$ & $Y$-invariant.

**Proof.** Just check axioms.
Let $X, Y : V \to V$ be commuting semisimple operators. Let $E_{\lambda_1}^X, \ldots, E_{\lambda_r}^X$ and $E_{\mu_1}^Y, \ldots, E_{\mu_s}^Y$ be the $e$-spaces of $X$ & $Y$ resp. Then

$$V = \bigoplus_{i=1}^r \bigoplus_{j=1}^s E_{\lambda_i}^X \cap E_{\mu_j}^Y$$

In particular, if $C : \mathbb{F}^n \to V$ is some co-ord system adapted to this direct sum decomposition, we may simultaneously diagonalise $X$ & $Y$ in the sense that both the representing matrices $C^{-1} \circ X \circ C$ & $C^{-1} \circ Y \circ C$ are diagonal.

**Proof.** By the propn, we know that $E_{\lambda_i}^X$ is $Y$-invariant so by the propn-defn, $Y$ restricted to $E_{\lambda_i}^X$ is semisimple & hence $E_{\lambda_i}^X$ is a direct sum of e-spaces wrt $Y$. But these e-spaces are just $E_{\lambda_i}^X \cap E_{\mu_j}^Y$. Hence

$$V = \bigoplus_{i=1}^r E_{\lambda_i}^X = \bigoplus_{i=1}^r \bigoplus_{j=1}^s E_{\lambda_i}^X \cap E_{\mu_j}^Y.$$ 

Simultaneous diagn follows from the fact that $E_{\lambda_i}^X \cap E_{\mu_j}^Y$ is both $X$-invariant & $Y$-invariant.
Assume from now on that $\mathbb{F} = \mathbb{C}$ & $V$ is an inner product space.

**Defn**

A linear map $T : V \rightarrow V$ is *normal* if $T^*$ exists & $T \circ T^* = T^* \circ T$. A matrix $A \in M_{nn}(\mathbb{C})$ is *normal* if $A^* A = AA^*$.

**E.g. 1** Any hermitian operator $T$ is normal. Why?

**E.g. 2** Any unitary operator $U : V \rightarrow V$ is also normal since
Spectral theorem for normal operators

**Theorem**

Let $T : V \rightarrow V$ be a linear map on a fin dim inner product space $V$.

1. If $T$ is normal, then there is an orthonormal co-ord system $U : \mathbb{C}^n \rightarrow V$ such that $U^{-1} \circ T \circ U$ is diagonal.

2. Conversely, if there is an orthonormal co-ord system $U : \mathbb{C}^n \rightarrow V$ such that $U^{-1} \circ T \circ U$ is diagonal, then $T$ is normal.

In particular, a complex matrix $A \in M_{nn}(\mathbb{C})$ is unitarily diagonalisable iff it is normal.

**Proof.** We prove 2) first. Let $D$ be the diagonal “matrix” $D = U^{-1} \circ T \circ U = U^* \circ T \circ U$. Note that $D^*$ is also a diagonal matrix so $DD^* = D^* D$. Hence

$$T \circ T^* = U \circ D \circ U^* \circ (U \circ D \circ U^*)^* = U \circ D \circ U^* \circ U \circ D^* \circ U^*$$

$$= U \circ D \circ D^* \circ U^* = U \circ D^* \circ D \circ U^* = T^* \circ T$$

We prove the converse after noting
Lemma

Let $T : V \rightarrow V$ be a linear map where $V$ is a fin dim inner product space $\mathbb{C}$. Let $X = \frac{1}{2}(T + T^*)$, $Y = \frac{1}{2i}(T - T^*)$. Then

1. both $X$, $Y$ are hermitian operators such that $T = X + iY$.  
2. If $T$ is normal then $X$, $Y$ commute.

Proof. This is an easy calculation.

We return to the proof of thm 1). By the spectral thm for self-adjoint operators, $X$, $Y$ are semisimple. Furthermore, if their respective e-spaces are $E^X_{\lambda_1}, \ldots, E^X_{\lambda_r}$ & $E^Y_{\mu_1}, \ldots, E^Y_{\mu_s}$, then the $E^X_{\lambda_i}$'s are mutually orthogonal as are the $E^Y_{\mu_j}$'s. The thm on simultaneous diag shows that we can find an orthonormal co-ord system $U : \mathbb{C}^n \rightarrow V$ adapted to the orthog direct sum decomp

$$V = \bigoplus_{i=1}^r \bigoplus_{j=1}^s E^X_{\lambda_i} \cap E^Y_{\mu_j}$$

such that both $U^{-1} \circ X \circ U$, $U^{-1} \circ Y \circ U$ are both diagonal. Then

$$U^{-1} \circ T \circ U = U^{-1} \circ (X + iY) \circ U = U^{-1} \circ X \circ U + i(U^{-1} \circ Y \circ U)$$

is also diagonal.
E.g. Show that \( A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \) is normal and unitarily diagonalise it.