Aim lecture: We examine the notion of isomorphisms for inner product spaces.

In this lecture, we let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $V, W$ be inner product spaces with inner products denoted $(\cdot|\cdot)$.

**Prop-Defn**

A linear map $T : V \rightarrow W$ preserves inner products if for all $v, v' \in V$ we have $(Tv|Tv') = (v|v')$. In this case $T$ is injective. If $T$ is also bijective, then we say that $T$ is an *isomorphism of inner product spaces*. In this case $T^{-1}$ also preserves inner products. A co-ordinate system $C : \mathbb{F}^n \rightarrow V$ is said to be *orthonormal* if $C$ is an isomorphism of inner product spaces.

**Proof.** Suppose $T$ preserves inner products & $v \in \ker T$. Then

$$0 = (Tv|Tv) = (v|v)$$

so $v = 0$ which shows $T$ to be injective.

It is an easy ex to see $T^{-1}$ preserves inner products.
Prop

Let \( C = (v_1 \ldots v_n) : \mathbb{F}^n \rightarrow V \) be linear. Then \( C \) preserves inner products iff \( B = \{v_1, \ldots, v_n\} \) is orthonormal. In particular, \( C \) is an orthonormal co-ordinate system iff \( B \) is an orthonormal basis for \( V \).

Proof. Suppose first that \( B \) is orthonormal & let \( v = (\beta_1, \ldots, \beta_n)^T, v' = (\beta'_1, \ldots, \beta'_n)^T \in \mathbb{F}^n \). Then

\[
(Cv|Cv') = (\sum_i \beta_i v_i | \sum_j \beta'_j v_j) = \sum_{i,j} \overline{\beta_i} \beta'_j (v_i | v_j) = \sum_i \overline{\beta_i} \beta'_i = (v|v')
\]

Thus \( C \) preserves inner products. The converse is an easy ex reversing the above computation.

E.g. Consider the \( \mathbb{R} \)-space \( V = \text{Span}(\sin x, \cos x) \) with inner product \( (f|g) = \int_{-\pi}^{\pi} f(t)g(t)dt \). Then the co-ord system \( C = \frac{1}{\sqrt{\pi}} (\sin x \quad \cos x) : \mathbb{R}^2 \rightarrow V \) is orthonormal.

Rem Every fin dim inner product space has an orthonormal co-ord system by Gram-Schmidt.
Isometries

**Defn**

Let $T : V \rightarrow W$ be a (not necessarily linear) fn. We say $T$ is an *isometry* of $V$ if it preserves “distances” in the sense that $\| T \mathbf{v} - T \mathbf{v}' \| = \| \mathbf{v} - \mathbf{v}' \|$ for all $\mathbf{v}, \mathbf{v}' \in V$.

**E.g. 1** Pick $\mathbf{v}_0 \in \mathbb{R}^n$. The translation by $\mathbf{v}_0$ map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n : \mathbf{v} \mapsto \mathbf{v} + \mathbf{v}_0$ is an isometry which is not linear unless $\mathbf{v}_0 = \mathbf{0}$.

*Why?*

**E.g. 2** Reflection about a plane in $\mathbb{R}^3$ is an isometry of $\mathbb{R}^3$ which is linear if the plane is a subspace.

**E.g. 3** Rotation about a line in $\mathbb{R}^3$ is an isometry of $\mathbb{R}^3$ which is linear if the line is a subspace.
Suppose that the inner product space $V$ is over $\mathbb{R}$

1. $(v|v') = \frac{1}{2} (\|v + v'\|^2 - \|v\|^2 - \|v'\|^2)$ for all $v, v' \in V$.

2. A linear map $T : V \to W$ preserves inner products iff $T$ is a linear isometry.

**Proof.** 1) Just calculate.

2) If $T$ preserves inner products then

$$\|Tv - Tv'\|^2 = (T(v - v')|T(v - v')) = (v - v'|v - v') = \|v - v'\|^2$$

so $T$ is an isometry. Conversely, if $T$ is a linear isometry, then

$$\|Tv\| = \|Tv - T0\| = \|v - 0\| = \|v\| \text{ for all } v \in V \text{ so } T \text{ preserves inner products by 1).}$$

**Rem** The result 2) above is true over $\mathbb{C}$ but you need a more complicated version of 1).
Unitary operators

Prop-Defn

Let $U : V \rightarrow W$ be linear. The following condns on $U$ are equiv.

1. $U$ is an isomorphism of inner product spaces.
2. $U^* \text{ exists} \& U^*U = \text{id}_V, \ \ UU^* = \text{id}_W$ (i.e. $U^* = U^{-1}$).

We say $U : V \rightarrow V$ is unitary or a unitary operator if $U^* = U^{-1}$. A complex matrix $A \in M_{nn}(\mathbb{C})$ is unitary if $A^* = A^{-1}$. A real matrix $A \in M_{nn}(\mathbb{C})$ is orthogonal if $A^T = A^{-1}$. In both the complex & real case, a matrix is unitary, resp orthogonal iff the columns are an orthonormal basis of $\mathbb{C}^n$, resp $\mathbb{R}^n$.

Proof. We need only show 1) $\iff$ 2). Suppose first that $U$ is an isomorphism of inner product spaces. Then 2) will follow if we can show $U^{-1}$ is the adjoint. But for all $v \in V, w \in W$ we have

$$(U^{-1}w|v) = (UU^{-1}w|Uv) = (w|Uv)$$

as required.

Suppose now that $U^{-1}$ is an adjoint of $U$. We need to show 1) holds. Indeed,
Example

**E.g.** Show that the real matrix \( R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \) is orthogonal.

**Rem** Geometrically, we knew this since \( R \) represents
The unitary group

Let $U_n$ denote the set of all unitary matrices in $M_{nn}(\mathbb{C})$

**Prop-Defn**

1. If $U, U' \in U_n$ then $UU' \in U_n$.
2. If $U \in U_n$ then $U^{-1} \in U_n$.

In particular, $U_n$ is a group when endowed with matrix multiplication. It is called the unitary group & has group identity $I_n$.

**Proof.** This follows from the following easy ex

**Lemma**

*If $T : V \rightarrow W, S : W \rightarrow X$ are linear maps between inner product spaces preserving inner products, then $S \circ T$ also preserves inner products.*

Similarly, the set of all orthogonal matrices in $M_{nn}(\mathbb{R})$ forms a group under matrix multn called the orthogonal group $O_n$. 
Unitarily similar matrices

**Defn**

Two matrices $A, B \in M_{nn}(\mathbb{C})$ are *unitarily similar* if there is a unitary matrix $U \in M_{nn}(\mathbb{C})$ such that $A = U^{-1}BU$. In this case $A = U^*BU$. Two matrices $A, B \in M_{nn}(\mathbb{R})$ are *orthogonally similar* if there is an orthogonal matrix $U \in M_{nn}(\mathbb{R})$ such that $A = U^{-1}BU$. In this case $A = U^TBU$.

If $B \in M_{nn}(\mathbb{C})$ & $A$ is some matrix representing it wrt some orthonormal change of co-ord systems, then $A$ & $B$ are unitarily similar.

**Prop**

Let $U : W \rightarrow W$ be linear & $T : V \rightarrow W$ be an isomorphism of inner product spaces. Then $(T^*UT)^* = T^*U^*T$. In particular, if $U$ is unitary, so is $T^*UT : V \rightarrow V$.

**Proof.** Indeed we just compute
Basic properties of unitary matrices

Prop

Let \( U \in U_n \).

1. \( |\det U| = 1. \)
2. If \( U \) is orthogonal then \( \det(U) = \pm 1 \)
3. The e-values of \( U \) also have modulus 1.

Proof. 1) & 2) Just note that

\[
1 = \det I = \det(UU^*) = \det(U)\det(U^T)
\]
\[
= \det(U)\det(U) = \det(U)\overline{\det(U)} = |\det(U)|^2.
\]

3) Let \( v \) be an e-vector with e-value \( \lambda \). Then

\[
\lambda(v|v) = (v|\lambda v) = (v|Uv) = (U^*v|v) = (U^{-1}v|v) = (\lambda^{-1}v|v) = \overline{\lambda^{-1}}(v|v).
\]

Hence \( \overline{\lambda} = \lambda^{-1} \) & \( |\lambda|^2 = 1 \) which gives the result.