Definition of adjoint

**Aim lecture:** We generalise the adjoint of complex matrices to linear maps between fin dim inner product spaces.

In this lecture, we let \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \). Let \( V, W \) be inner product spaces with inner products denoted \( (\cdot|\cdot)_V, (\cdot|\cdot)_W \). Let \( D_V : V \to V^*, D_W : W \to W^* \) be the canonical maps.

**Prop-Defn**

Let \( T : V \to W, T' : W \to V \) be linear.

1. \( T^T \circ D_W = D_V \circ T' : W \to V^* \) iff for all \( v \in V, w \in W \) we have \( (T'w|v)_V = (w|Tv)_W \).

2. Given \( T \), there is at most one linear map \( T' : W \to V \) satisfying the condn in 1) above. In this case we write \( T' = T^* \) & call it the adjoint of \( T \).

3. If \( V \) is fin dim then \( T^* \) exists & \( T^* = D_V^{-1} \circ T^T \circ D_W \).

**Rem** The defn of the adjoint depends critically on the inner products involved though the notation does not record this!

**Proof.** For 3), just note that \( D_V \) is invertible & \( D_V^{-1} \circ T^T \circ D_W \) is linear being the composite of two conjugate linear maps with a linear one.
Proof cont’d

For 1) just note

\[ T^T \circ D_W = D_V \circ T' \iff \text{for all } w \in W, \ T^T(D_Ww) = D_V(T'w) \]
\[ \iff \text{for all } w \in W, \ (D_Ww) \circ T = (T'w|\cdot) \]
\[ \iff \text{for all } w \in W, \ (w|T(\cdot)) = (T'w|\cdot) \]

For 2), suppose \( T', T'' \) both satisfy the condn in 1). Then \( D_V \circ T' = D_V \circ T'' \). Suffice show for any \( w \in W \) that \( T'w = T''w \). But \( D_V(T'w) = D_V(T''w) \) so the result follows from injectivity of \( D_V \).
**E.g.** Let $V = C^\infty_c$ be the $\mathbb{R}$-space of compactly supported (= zero outside of some compact set) infinitely differentiable functions on $\mathbb{R}$. Note $T = \frac{d}{dt}$ is a linear map from $V \rightarrow V$. Note that we have an inner product $(f|g) = \int_{-\infty}^{\infty} f(t)g(t)dt$. What's $T^*$?

**A** Given $f, g \in V$ note that integration by parts gives

$$
(f|\frac{dg}{dt}) = \int_{-\infty}^{\infty} f(t)\frac{dg(t)}{dt} dt
$$

$$
= [f(t)g(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df(t)}{dt} g(t) dt
$$

$$
= -\int_{-\infty}^{\infty} \frac{df(t)}{dt} g(t) dt
$$

$$
= (\frac{df}{dt}|g)
$$

Hence $T^* = -\frac{d}{dt}$. 

Q Let $A \in M_{mn}(\mathbb{F})$ & $T_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ be the assoc lin map. What is $T_A^* : \mathbb{F}^m \rightarrow \mathbb{F}^n$?

Answer

$T_A^* = T_A^*$

Proof. It suffices to show that for any $\mathbf{v} \in \mathbb{F}^n$, $\mathbf{w} \in \mathbb{F}^m$ we have $(A^* \mathbf{w} | \mathbf{v}) = (\mathbf{w} | A \mathbf{v})$. Indeed
The following formulae hold whenever they make sense. $S, T$ are appropriate linear maps.

1. $(S + T)^* = S^* + T^*$.
2. $(\beta S)^* = \overline{\beta} S^*$ for $\beta \in \mathbb{F}$.
3. $(S \circ T)^* = T^* \circ S^*$.
4. $(T^*)^* = T$.
5. $id^* = id$.

In particular, if $V, W$ are fin dim, then the adjoint operator $(\cdot)^* : L(V, W) \to L(W, V) : S \mapsto S^*$ is conjugate linear.

**Proof.** These all follow from defns & corresponding results for the transpose. For example,
Orthogonal complements to kernels

Prop

Let $T : V \rightarrow W$ be a linear map between fin dim inner product spaces. Then

1. $(\ker T)^\perp = \text{im } T^*$
2. If $V, W$ are fin dim then, $\text{rank } T = \text{rank } T^*$.

Proof. 1) We show equiv that $(\text{im } T^*)^\perp = \ker T$. Let $v \in V$

$$v \in (\text{im } T^*)^\perp \iff (T^*w|v) = 0 \text{ for all } w \in W$$
$$\iff (w|Tv) = 0 \text{ for all } w \in W$$
$$\iff 0 = (Tv|\cdot) = D_W(Tv)$$
$$\iff Tv = 0, \text{ (recall } D_W \text{ is injective})$$
$$\iff v \in \ker T$$

For 2) we use rank-nullity

$$\text{rank } T^* = \dim(\text{im } T^*) = \dim((\ker T)^\perp)$$
$$= \dim V - \dim(\ker T) = \dim(\text{im } T) = \text{rank } T.$$
Kernel of $T^* \circ T$

**Prop**

Let $T : V \rightarrow W$ be linear.

1. If $S : W \rightarrow X$ is linear then $\ker(S \circ T) \supseteq \ker T$.
2. $\ker T^* \circ T = \ker T$.
3. If $V, W$ are fin dim then $\text{rank } T^* \circ T = \text{rank } T$.

**Proof.** 1) is easy ex.

2) By 1), it suffices to show $\ker T^* \circ T \subseteq \ker T$ so let $v \in \ker T^* \circ T$. Then

$$0 = (T^*(Tv)|v) = (Tv|Tv)$$

so $Tv = 0$ & $v \in \ker T$. Part 2) follows.

3) This follows from the rank-nullity thm since part 2) ensures the nullities are the same (whilst the domains are also the same).