**Aim lecture:** We generalise the notion of transposes of matrices to arbitrary linear maps by introducing dual vector spaces.

In most of this lecture, we allow $\mathbb{F}$ to be a general field.

**Defn**

Let $V = \mathbb{F}$-space. The *dual* of $V$ is the $\mathbb{F}$-space $V^* = L(V, \mathbb{F})$. The elements of $V^*$ are called *linear functionals*.

**E.g. 1** $(\mathbb{F}^n)^* = L(\mathbb{F}^n, \mathbb{F}) = M_{1n}(\mathbb{F})$, the set of length $n$ row vectors in $\mathbb{F}$. Note that the transpose map $(\cdot)^T : \mathbb{F}^n \longrightarrow (\mathbb{F}^n)^* : v \mapsto v^T & \mathbb{F}^n$ is an isomorphism!

**E.g. 2** If $X$ is a set & $V = \text{Fun}(X, \mathbb{F})$, then for every $x \in X$ we obtain an element $ev_x \in V^*$.

**E.g. 3** Given an inner product $(\cdot|\cdot)$ on $V$ & $v \in V$ we have $(v|\cdot) \in V^*$. 
Visualising linear functionals on $\mathbb{R}^n$

Let $l \in (\mathbb{R}^n)^*$ is in particular, a fn $l : \mathbb{R}^n \to \mathbb{R}$ so can be visualised by drawing level curves. Suppose $l \neq 0$ so $\text{im } l = \mathbb{R}$.

The level curve $l = 0$ is just $\ker l$ which is a subspace of dim...
We can now generalise the transpose map for matrices.

**Prop-Defn**

Let $S : V \rightarrow W$ be linear. The *transpose* of $S$ is the fn $S^T : W^* \rightarrow V^*$ defined by $S^T f = f \circ S : V \xrightarrow{S} W \xrightarrow{f} \mathbb{F}$ for $f \in W^*$. $S^T$ is linear.

**Proof.** Linearity follows from the distributive law: for $f, g \in W^*$ we have $(f + g) \circ S = f \circ S + g \circ S$ & the fact that $(\beta f) \circ S = \beta(f \circ S)$ for $\beta \in \mathbb{F}$.

**E.g.** Let $S = \frac{d}{dx} : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ so $S^T : \mathbb{R}[x]^* \rightarrow \mathbb{R}[x]^*$. What's $S^T \text{ev}_0 \in \mathbb{R}[x]^*$?
Connection with matrix transpose

Q Let $A \in M_{mn}(\mathbb{F})$ & $T_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ be the assoc lin map. What is $T_A^T : (\mathbb{F}^m)^* = M_{1m}(\mathbb{F}) \rightarrow (\mathbb{F}^n)^* = M_{1n}(\mathbb{F})$?

Simplest answer $T_A^T$ is pre-composition (= right composition) with $T_A$. Since composition corresponds with matrix multn, this is right multn by $A$. Note, $T_A$ is left multn by $A$.

More formally, let $v^T \in M_{1m}(\mathbb{F}) = (\mathbb{F}^m)^*$, $w \in \mathbb{F}^n$. Then

$$(T_A^T v^T)w = (v^T \circ T_A)w = (v^T A)w$$

so $T_A^T v^T = v^T A$.

Q What’s the matrix representing $T_A^T$ wrt natural co-ordinate systems $(\cdot)^T : \mathbb{F}^m \rightarrow (\mathbb{F}^m)^*$, $(\cdot)^T : \mathbb{F}^n \rightarrow (\mathbb{F}^n)^*$.

Answer

The matrix representing $T_A^T$ wrt the natural co-ord systems is $A^T$.

Why?

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Functoriality

Prop

1. \((\text{id}_V)^T = \text{id}_{V^*} : V^* \to V^*\).

2. Consider a composite of linear maps \(S \circ T : U \xrightarrow{T} V \xrightarrow{S} W\). Then \((S \circ T)^T = T^T \circ S^T : W^* \to U^*\).

3. In particular, if \(S : V \to W\) is an isomorphism, so is \(S^T\).

Proof. For 1) note \((\text{id}_V)^T : f \mapsto f \circ \text{id}_V = f\).

2) For \(f \in W^*\), we have

3) Suffice show \(S^{-T} = (S^{-1})^T\) is inverse to \(S^T\). But by 1) & 2)

\[ S^T \circ (S^{-1})^T = (S^{-1} \circ S)^T = \text{id}^T = \text{id} \]

& \(\text{sim } (S^{-1})^T \circ S^T = \text{id}\) so we are done.
Prop

Let $V = \text{fin dim } \mathbb{F}$-space. Then $\dim V^* = \dim V$.

Proof. Let $C : \mathbb{F}^d \longrightarrow V$ be a co-ord system so $\dim = d$. Then $C^T : V^* \longrightarrow (\mathbb{F}^d)^*$ is also an isomorphism so

$$\dim V^* = \dim (\mathbb{F}^d)^* = d.$$ 

E.g. $\{\text{ev}_0, \text{ev}_1\}$ is a basis for $\mathbb{C}[x]_{\leq 1}^*$ since it is lin indep (ex) & $\dim \mathbb{C}[x]_{\leq 1}^* = \dim \mathbb{C}[x]_{\leq 1} = 2$. 


The matrix transpose operator \((\cdot)^T : M_{mn}(\mathbb{F}) \rightarrow M_{nm}(\mathbb{F}) : A \mapsto A^T\) is linear.

This suggests

**Prop**

Let \(\beta \in \mathbb{F} \& R, S : V \rightarrow W\) be linear. Then

1. \((R + S)^T = R^T + S^T\)
2. \((\beta S)^T = \beta S^T\)

In particular the map \((\cdot)^T : L(V, W) \rightarrow L(W^*, V^*) : S \mapsto S^T\) is linear.

**Proof.** For \(f \in W^*\), this follows from
Let now $V = \mathbb{F}$-space equipped with an inner product $(\cdot|\cdot)$ so $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

**Prop**

1. The canonical map $D : V \rightarrow V^* : v \mapsto (v|\cdot)$ is an injective conjugate linear map.
2. If $V$ is fin dim then $D$ is a conjugate linear isomorphism.

**Proof.**

1) (Ex) Conjugate linearity of $(\cdot|w)$ shows $D$ is conjugate linear. To check injectivity suppose $Dv = Dv'$ so that by conjugate linearity

$$(v - v'|v - v') = (v|v - v') - (v'|v - v') = (Dv)(v - v') - (Dv')(v - v') = 0$$

so $v - v' = 0$ by inner product axioms.

2) We now check $D$ is onto & show any $l \in V^*$ is in the image of $D$. Now $D0 = 0$ so we can assume $l \neq 0$ so has image $\mathbb{F}$. Our isomorphism thm applied to $l : V \rightarrow \mathbb{F} \rightarrow \mathbb{F}$ for any vector space complement $W$ to ker $l$ is isomorphic to $\mathbb{F}$ via the restricted map $l|_W : W \rightarrow \mathbb{F}$. This holds in particular for $W = (\ker l)^\perp$ so we can find $w \in W$ with $l(w) = 1$. 
Proof completed

It suffices now to show $D \frac{w}{\|w\|^2} = l$. Now any vector in $V = W \oplus (\ker l)$ can be written in the form $\beta w + v$ with $\beta \in \mathbb{F}, v \in \ker l$. Then

$$
\left(D \frac{w}{\|w\|^2}\right)(\beta w + v) = \left(\frac{w}{\|w\|^2}\right)\beta w + v = \beta \left(\frac{w}{\|w\|^2}\right)w = \beta = l(\beta w + v)
$$

& we are done.

**E.g.** Let $V = \mathbb{R}[x]_{\leq 1}$ with inner product $(f|g) = \int_0^1 f(t)g(t)dt$. Find $f \in V$ such that $(f|\cdot) = ev_0$. 