**Aim lecture:** Vector spaces have some geometry but their data encodes no info about angles & lengths. For this we need inner products which we define here.

Throughout this lecture (& the next few) $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

**Defn**

Let $V = \mathbb{F}$-space. An inner product on $V$ is a function $(\cdot|\cdot) : V \times V \longrightarrow \mathbb{F}$ : $(v, w) \mapsto (v|w)$ such that for all $v, w, w' \in V, \beta \in \mathbb{F}$ we have

1. $(v|\beta w + w') = \beta (v|w) + (v|w')$
2. $(w|v) = (\overline{v|w})$ where the bar denotes complex conjugation.
3. $(v|v)$ is a positive real if $v \neq 0$.

An *inner product space* is a vector space $V$ equipped with an inner product as above.

**Rem 1** Axiom 1) means that the function $l_v = (v|\cdot) : V \longrightarrow \mathbb{F}$ : $w \mapsto (v|w)$ is linear so in particular $(v|0) = 0$.

**Rem 2** If $\mathbb{F} = \mathbb{R}$ then axiom 2) simplifies to $(w|v) = (v|w)$

**E.g.** For $V = \mathbb{R}^n$, the dot product $(v|w) = v \cdot w = v^T w$ is an inner product on $V$. 
Conjugates & adjoints of complex matrices

Let $A = (a_{ij})_{ij} \in M_{mn}(\mathbb{C})$. The \textit{conjugate} of $A$ is $\overline{A} = (\overline{a_{ij}})_{ij}$. The \textit{adjoint} of $A$ is $A^* = \overline{A}^T = \overline{A^T}$. (Do not confuse with the classical adjoint!).

**Prop**

Let $A, B \in M_{mn}(\mathbb{C}), X \in M_{nl}(\mathbb{C}), \beta \in \mathbb{C}$. Then

1. $\overline{(A + B)} = \overline{A} + \overline{B}$
2. $\overline{\beta A} = \overline{\beta} \overline{A}$
3. $\overline{AX} = \overline{A} \overline{X}$
4. $(A + B)^* = A^* + B^*$
5. $(\beta A)^* = \overline{\beta} A^*$
6. $(AX)^* = X^* A^*$
7. $A^{**} = A$

**Proof.** These are good easy ex. For example
Standard inner product on $\mathbb{C}^n$

**Prop**

The following is an inner product on the $\mathbb{C}$-space $\mathbb{C}^n$

$$(v|w) = v^* w.$$

It is called the *standard inner product* on $\mathbb{C}^n$.

**Proof.**
Fix an interval \([a, b] \subseteq \mathbb{R}\). Let \(V = \) the vector space of continuous \(F\)-valued functions on \([a, b]\).

**Defn**

The *standard* inner product on \(V\) is for \(f, g \in V\)

\[
(f | g) = \int_a^b f(t)g(t)\,dt
\]

**Proof.** Easy ex in checking axioms. For example,

**E.g.**
Prop

Let $V$ be a $\mathbb{F}$-space equipped with an inner product $\langle \cdot | \cdot \rangle$ & $W \leq V$. Then $\langle \cdot | \cdot \rangle$ restricts to an inner product on $W$ too.

**Proof.** Inner product axioms are immediate.

**E.g.** By identifying real & complex polys with the functions they induce on $[a, b]$ we get an inner product on $\mathbb{F}[x]$ or $\mathbb{F}[x]_{\leq d}$. 
Defn

Let $V$ be an inner product space with inner product $(\cdot|\cdot)$. The *norm* of $v \in V$ is $\|v\| = \sqrt{(v|v)}$. We say that $S, S' \subseteq V$ are orthogonal if $(v|v') = 0$ (so also $(v'|v) = 0$) for all $v \in S, v' \in S'$. In this case we write $S \perp S'$.

E.g. Let $V =$ the vector space of continuous $\mathbb{R}$-valued functions on $[0, 2\pi]$ equipped with the standard inner product.
Geometry in inner product spaces

Prop

Let $V$ be an inner product space with inner product $(\cdot | \cdot)$. Let $v, w \in V$.

1. (Cauchy-Schwarz inequality) $|(v | w)| \leq \|v\| \|w\|$ with equality iff $v, w$ are lin dep.

2. (Triangle inequality) $\|v + w\| \leq \|v\| + \|w\|$

3. (Pythagoras theorem) If $v \perp w$ then $\|v + w\|^2 = \|v\|^2 + \|w\|^2$.

Proof. Same as for dot products in $\mathbb{R}^n$ so we only prove 3)

E.g. $\left(\int_0^1 f(t)g(t)dt\right)^2 \leq \left(\int_0^1 |f(t)|^2 dt\right) \left(\int_0^1 |g(t)|^2 dt\right)$
by
Conjugate linearity

Prop-Defn

Let $T : V \rightarrow W$ be a fn between $\mathbb{F}$-spaces. We say that $T$ is conjugate linear if for $v, w \in V, \beta \in \mathbb{F}$ we have:

1. $T(v + w) = T(v) + T(w)$
2. $T(\beta v) = \overline{\beta} T(v)$.

1. The composite of conjugate linear maps is linear.
2. The composite of a conjugate linear map with a linear one (in either order that makes sense) is conjugate linear.
3. If $T$ is an invertible conjugate linear map, then $T^{-1}$ is also conjugate linear & we say $T$ is a conjugate linear isomorphism.

Proof. Easy ex.

E.g. 1 Given an inner product space $V$ the map $(\cdot | w) : V \rightarrow \mathbb{F} : v \mapsto (v|w)$ is conjugate linear.

E.g. 2 $V = \mathbb{C}^n$