Eigenvectors

**Aim lecture:** The simplest $T$-invariant subspaces are 1-dim & these give rise to the theory of eigenvectors. To compute these we introduce the similarity invariant, the characteristic polynomial.

**Prop-Defn**

Let $T : V \rightarrow V$ be linear. The following are equivalent condns on $v \in V$.

1. $Fv$ is a $T$-invariant (automatically 1-dimensional) subspace.
2. $Tv = \lambda v$ for some $\lambda \in F$.
3. $v \in \ker(T - \lambda \text{id})$ for some $\lambda \in F$.

If these hold & furthermore $v \neq 0$ we say $v$ is an *eigenvector* for $T$ with *eigenvalue* $\lambda$.

**Proof.** Clearly 2) $\iff$ 3) since

$$Tv = \lambda v \iff Tv - \lambda \text{id} v = 0 \iff v \in \ker(T - \lambda I).$$

2) $\implies$ 1) by lemma on testing invariance, lecture 20.

For 1) $\implies$ 2) note
Eigenvalues & eigenspaces of an endomorphism

**Defn**

Let $T : V \rightarrow V$ be linear. An eigenvalue of $T$ is a scalar $\lambda \in F$ such that there is an e-vector $v$ for $T$ with eigenvalue $\lambda$. Given such an e-value, the $\lambda$-eigenspace of $T$ is the subspace $E_\lambda = \ker(T - \lambda \text{id}) \leq V$. The geometric multiplicity of $\lambda$ is $\dim E_\lambda$.

**E.g.** Any real number $\lambda$ is an e-value for $T = \frac{d}{dx} : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ since

In fact $E_\lambda =$
**Characteristic polynomial of square matrices**

To find e-values of square matrices we need

**Prop-Defn**

Let \( A = (a_{ij}) \in M_{nn}(\mathbb{F}) \). The **characteristic polynomial of** \( A \) is the function \( cp_A(\lambda) = \det(A - \lambda I_n) \) where \( I_n \) is the \( n \times n \)-identity matrix. This function is a polynomial function of degree \( n \) with co-efficients in \( \mathbb{F} \).

**Proof.** Note that \( A - \lambda I_n = (a'_{ij}) \) where \( a'_{ij} = a_{ij} \) if \( i \neq j \) whilst \( a'_{ii} = a_{ii} - \lambda \). Now

\[
\det(A - \lambda I_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a'_{i \sigma(i)}
\]

which is clearly a polynomial function of \( \lambda \).

The summand corresponding to \( \sigma = \text{id} \) is \((a_{11} - \lambda)(a_{22} - \lambda)\ldots(a_{nn} - \lambda)\) which has degree \( n \).

Any other summand contains at least two non-diagonal entries so has degree \( \leq n - 2 \). Hence, \( \deg cp_A(\lambda) = n \).

**Scholium** The co-efficient of \( \lambda^{n-1} \) in \( cp_A(\lambda) \) is \((-1)^{n-1} \sum_i a_{ii} \).
Prop-defn

Let $A \in M_{nn}(\mathbb{F})$. Then $\lambda$ is an e-value for $A$ iff it is a root of the characteristic polynomial of $A$. In this case, the multiplicity of the root is called the \textit{algebraic multiplicity} of the e-value.

\textbf{Proof.} $\lambda$ is an e-value iff $A$ has an e-vector with e-value $\lambda$ iff $\ker(A - \lambda I_n) \neq 0$ iff $A - \lambda I_n$ is not invertible iff $\det(A - \lambda I_n) = 0$.

\textbf{E.g.} Find the e-values of $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ & their algebraic & geometric multiplicities.
Prop-Defn

Two matrices $A, B \in M_{nn}(\mathbb{F})$ are similar if there exists $C \in GL_n(\mathbb{F})$ such that $A = C^{-1}BC$ i.e. $A$ is a matrix representing $B$ wrt some co-ordinate system. Being similar is an equivalence relation. In particular,

1. if $A$ is similar to $B$ then $B$ is similar to $A$.
2. if $A$ is similar to $B$ and $B$ is similar to $D$, then $A$ is similar to $D$.

The set of all matrices similar to $A$ is called the similarity class of $A$.

Proof. Easy ex.

E.g. $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ are not similar for if $A = C^{-1}BC$ then
Similarity invariants

**Defn**

A function of the form \( f : M_{nn}(\mathbb{F}) \longrightarrow X \) is a *similarity invariant* if \( f(A) = f(B) \) whenever \( A, B \) are similar.

**E.g. 1** The characteristic polynomial \( cp : M_{nn}(\mathbb{F}) \longrightarrow \mathbb{F}[\lambda] : A \mapsto cp_A(\lambda) \) is a similarity invariant since if \( B = C^{-1}AC \) for some \( C \in GL_n(\mathbb{F}) \) then

\[
\det(B - \lambda I_n) = \det(C^{-1}AC - \lambda I_n) = \det(C^{-1}[A - \lambda I_n]C) =
\]

**E.g. 2** In particular, any of the co-efficients of the characteristic polynomial are similarity invariants. The important ones (up to sign) are the determinant \( \det(A) = cp_A(0) \) & the *trace* which is defined to be \( \text{tr}(A) = \sum_{i=1}^{n} a_{ii} \) where \( A = (a_{ij}) \).

**E.g. 3** Similarly, the set of eigenvalues is a similarity invariant.
Sums & products of e-values

Let $A \in M_{nn}(\mathbb{C})$. Since $\mathbb{C}$ is alg. closed, it has $n$ e-values $\lambda_1, \ldots, \lambda_n$ when counted with (algebraic) multiplicity.

**Formula**

$$\text{tr}(A) = \sum_i \lambda_i, \quad \text{det}(A) = \prod_i \lambda_i$$

**Proof.** Note the following equality of polynomials in $\lambda$

$$\text{det}(A - \lambda I_n) = \prod_i (\lambda_i - \lambda).$$

Equating constant terms gives $\text{det}(A) = \prod_i \lambda_i$ while comparing co-effs of $\lambda^{n-1}$ gives the trace formula.

**E.g.** Suppose you know two of the e-values of $A \in M_{33}(\mathbb{C})$ are 2, 3 and $A$ has diagonal entries 1, 1, 4. Find the third e-value $\lambda_3$. 

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Characteristic polynomials of endomorphisms

Similarity invariants can be extended to endomorphisms of finite dimensional vector spaces. For example

\[ V = \text{fin dim } F\text{-space} & T : V \rightarrow V \text{ be linear. For any co-ordinate system } C : F^n \rightarrow V, \text{ we may define the characteristic polynomial of } T, \text{ denoted } \text{cp}_T(\lambda), \text{ to be the characteristic polynomial of the representing matrix } C^{-1} \circ T \circ C. \text{ This is well-defined since given any other co-ordinate system } C_1 : F^n \rightarrow V, \text{ the characteristic polynomials of } C^{-1} \circ T \circ C & C_1^{-1} \circ T \circ C_1 \text{ are the same, so the definition is independent of the choice of co-ord system.} \]

**Proof.** We need only check equality of characteristic polynomials by showing \( C^{-1} \circ T \circ C & C_1^{-1} \circ T \circ C_1 \text{ are similar. Indeed} \)

\[
C_1^{-1} \circ T \circ C_1
\]

**Rem** We similarly can define \( \text{det}(T), \text{tr}(T) \) etc.

**E.g.** We have seen that \( T = \frac{d}{dx} : \mathbb{R} \cos x \oplus \mathbb{R} \sin x \rightarrow \mathbb{R} \cos x \oplus \mathbb{R} \sin x \text{ is} \)

represented by the matrix

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Using co-ordinates, we can calculate e-vectors, e-values & e-spaces using

Prop

Let $V = \text{fin dim } \mathbb{F}$-space & $T : V \rightarrow V$ be linear. Let $C : \mathbb{F}^n \rightarrow V$ be a co-ord system & $A = C^{-1} \circ T \circ C$ be the representing matrix. Then

1. $x \in \mathbb{F}^n$ is an e-vector of $A$ with e-value $\lambda$ iff $Cx$ is an e-vector of $T$ with e-value $\lambda$.

2. The e-values of $T$ & $A$ are the same. They are the roots of $\text{cp}_A(\lambda) = \text{cp}_T(\lambda)$.

3. If $E_\lambda$ is the $\lambda$-e-space of $A$ then $C(E_\lambda)$ is the $\lambda$-e-space of $T$.

Proof. We just prove 1), as 2) & 3) readily then follow.

$$Ax = \lambda x \iff C^{-1} \circ T \circ Cx = \lambda x \iff$$
Example

**E.g.** We compute the e-vectors & e-values of $T : \mathbb{R}[x]_{\leq 2} \rightarrow \mathbb{R}[x]_{\leq 2}$ defined by

$$(Tp)(x) = xp'(x) - 2p'(x) - p(x).$$