Philosophy of studying linear maps $T : V \rightarrow V$

**Aim lecture:** One of the main objects of study in linear algebra are *endomorphisms*, linear maps of the form $T : V \rightarrow V$ where the domain & co-domain are equal. The key is that there are some co-ordinate systems $C : \mathbb{F}^n \rightarrow V$ which make the study of $T$ easier & are somehow preferred by $T$. In this lecture we explain an approach for finding these preferred co-ordinate systems.

From a theoretical point of view, it is better to rephrase our question of finding good co-ordinate systems as follows. Note that a co-ordinate system is essentially a way of writing $V$ as a direct sum of copies of $\mathbb{F}$. So we may simplify our question & ask, can we decompose $V = V' \oplus V''$ for some subspaces $V'$, $V''$ in such a way as to better understand $T$. If so, we can try to repeat & further decompose $V'$ & $V''$.

If we're lucky, we eventually arrive at a decomposition of $V$ into a direct sum of 1-dimensional vector spaces & we have a co-ordinate system. If not we can hopefully still write (in a useful way) $V$ as a direct sum of subspaces $V_i$ of smaller dimension & we get a co-ordinate system by arbitrarily picking bases for each $V_i$. 
Generalisation of internal direct sums

We generalise the notion of internal direct sums to $\geq 2$ subspaces, first inductively & then by relating to the (external) direct sum.

Prop-Defn

Let $V_1, \ldots, V_r \subseteq V$. We say the sum $V_1 + \ldots + V_r$ is direct if the sum of $r - 1$ subspaces $V_1 + \ldots + V_{r-1}$ is direct, and also the sum of 2 subspaces $(V_1 + \ldots + V_{r-1}) + V_r$ is direct. This is equivalent to saying the natural linear map

$$\Phi = (\text{id} \mid_{V_1} \ldots \mid_{V_r}) : V_1 \oplus \ldots \oplus V_r \longrightarrow V_1 + \ldots + V_r : (v_1, \ldots, v_r)^T \mapsto v_1 + \ldots + v_r$$

is an isomorphism. In this case, we will often write (abusing notation) the internal direct sum $V_1 + \ldots + V_r$ as $V_1 \oplus \ldots \oplus V_r$. Furthermore, for $v \in \sum_i V_i$ we have $\Phi^{-1}(v) = (v_1, \ldots, v_r)^T$ where $v = v_1 + \ldots + v_r$ is the unique way of expressing $v$ as a sum with $v_1 \in V_1, \ldots, v_r \in V_r$.

Proof. This is by induction using the case of two subspaces.

E.g. $\mathbb{F}^3$ is the internal direct sum of the 3 subspaces $\mathbb{F} e_1, \mathbb{F} e_2, \mathbb{F} e_3$. 
Definition of invariance

Given $T : V \rightarrow V$ linear, the good way of decomposing $V$ into an internal direct sum of subspaces is given in

Defn

A subspace $V' \leq V$ is $T$-invariant or invariant wrt $T$ if $T(V') \subseteq V'$. In this case, $T$ restricts to a linear endomorphism $T|_{V'} : V' \rightarrow V'$. Suppose that $V = V_1 \oplus \ldots \oplus V_r$ is the internal direct sum of subspaces. We say the internal direct sum is $T$-invariant or invariant wrt $T$ if each subspace $V_1, \ldots, V_r$ is $T$-invariant.

E.g. Let $T : \mathbb{F}^3 \rightarrow \mathbb{F}^3$ be the linear map given by the matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 3 \\
0 & 4 & 5
\end{pmatrix}.
\]

The subspace $V' = \mathbb{F}(1, 1, 0)^T$ is not invariant since
Note that \( \mathbb{F}^3 \) is the internal direct sum of the subspaces 
\( V' = \mathbb{F} e_1, \; V'' = \mathbb{F} e_2 + \mathbb{F} e_3 \). Moreover \( V', \; V'' \) are \( T \)-invariant since 

Hence \( \mathbb{F}^3 \) is a \( T \)-invariant direct sum of \( V' \) & \( V'' \).
Example of 3-dim rotation

Let \( \mathbf{v} \in \mathbb{R} \) be a unit vector & \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be rotation about the line \( L = \mathbb{R} \mathbf{v} \). \( T \) is a linear map (as can be checked geometrically). Also \( \mathbb{R}^3 \) is the \( T \)-invariant direct sum of \( L \) & \( P = \)

**Rem** Geometrically it is obvious that the best co-ordinates to use for \( T \) is to make \( L \) one of the co-ordinate axes & have the other two axes on \( P \). The general principle why this is true is the \( T \)-invariance of the direct sum.
Prop-Defn

Suppose that $V$ is the internal direct sum of $V_1$ & $V_2$. The *projection onto* $V_1$ (wrt the direct sum $V_1 \oplus V_2$) is the linear map $P: V \to V$ corresponding the map on external direct sums $\tilde{P}: V_1 \oplus V_2 \to V_1 \oplus V_2$ given by the $2 \times 2$-matrix in $(L(V_j, V_i))_{ij}$

\[
\begin{pmatrix}
\text{id} & V_1 \\
0 & 0
\end{pmatrix}.
\]

That is, $P = \Phi \circ \tilde{P} \circ \Phi^{-1}: V \to V_1 \oplus V_2 \to V_1 \oplus V_2 \to V$ where $\Phi: V_1 \oplus V_2 \to V: (v_1 \ v_2) \mapsto v_1 + v_2$ is the natural isomorphism.

1. More explicitly $Pv = v_1$ where we have written uniquely $v = v_1 + v_2$ with $v_1 \in V_1, v_2 \in V_2$.

2. In this case, $V = V_1 \oplus V_2$ is a $P$-invariant direct sum.

Proof is an easy ex. Check 1) first.

2) is also easy, but it’s more instructive to look at a geometric

E.g. $V = \mathbb{R}^2 = \mathbb{R}v \oplus \mathbb{R}w$ for non-parallel vectors $v, w$. 
Prop

Let $T : V \rightarrow V$ be linear & $V = V_1 + \ldots + V_r$ be a $T$-invariant internal direct sum. Let $\Phi : V_1 \oplus \ldots \oplus V_r \rightarrow V$ be the natural isomorphism with the external direct sum. Then $\Phi^{-1} \circ T \circ \Phi = T|_{V_1} \oplus \ldots \oplus T|_{V_r}$ as maps from $V_1 \oplus \ldots \oplus V_r \rightarrow V_1 \oplus \ldots \oplus V_r$. Alternatively, this means that (on identifying internal & external direct sums) we may write $T$ as a matrix in $(L(V_j, V_i))$ with block diagonal form

$$
\begin{pmatrix}
T|_{V_1} & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & T|_{V_r}
\end{pmatrix}.
$$

where we consider $T|_{V_i} : V_i \rightarrow V_i$ as usual.

Proof. For ease of writing, we do the case $r = 2$. Recall the natural isomorphism $\Phi : V_1 \oplus V_2 \rightarrow V_1 + V_2 : \binom{v_1}{v_2} \mapsto v_1 + v_2.$
We know that $\Phi^{-1} \circ T \circ \Phi = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$. For $v_1 \in V_1$ we have

$$\begin{pmatrix} T v_1 \\ 0 \end{pmatrix} = \Phi^{-1}(Tv_1) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \Phi^{-1}v_1 = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} T_{11}v_1 \\ T_{21}v_1 \end{pmatrix}.$$

Comparing LHS & RHS we see $T_{11} = T|_{V_1}$, $T_{21} = 0$. Similarly, $T_{22} = T|_{V_2}$, $T_{12} = 0$. 