Isomorphisms preserve linear concepts

**Aim lecture:** Given a vector space or linear map between vector space, the easiest way to compute with them is to use co-ordinate systems. We show how this works.

**Prop 1**

Let $T : V \longrightarrow W$ be an isomorphism of $\mathbb{F}$-spaces & $B \subseteq V$ be a finite subset.

1. For any subspace $V' \leq V$, we obtain an isomorphism by restriction $T|_{V'} : V' \longrightarrow T(V')$.
2. $B$ is a spanning set for $V$ iff $T(B)$ is a spanning set for $W$.
3. $B$ is linearly independent iff $T(B)$ is.
4. $B$ is a basis for $V$ iff $T(B)$ is basis for $W$.

**Rem** An important case where we use this propn is where $T$ is a co-ordinate system.

**Proof.** For 1) note that $T|_{V'}$ is surjective because we altered the co-domain to $T(V')$. It is injective since the eqn $Tv = w$ still has unique solns (if at all). Finally, we know $T|_{V'}$ is linear so we are done.
Proof continued

Note that 4) follows from 2) & 3) put together.

2) (\Rightarrow) Suppose \( B = \{v_1, \ldots, v_n\} \) spans \( V \). Hence for any \( w \in W \) we may write \( T^{-1}w = \sum_i \beta_i v_i \) for some scalars \( \beta_i \in F \). Then \( w = \sum_i \beta_i T v_i \in \text{Span}(T(B)) \) so the forward implication holds. To prove the reverse implication, we just apply the (\Rightarrow) result proved to \( T^{-1} : W \rightarrow V \) & the subset \( T(B) \).

3) (\Rightarrow) We prove the contrapositive & suppose \( T(B) \) is linearly dependent so there is a non-trivial linear relation

\[
\sum_i \beta_i T v_i = 0 \quad \text{for some } \beta_i \in F
\]

say with \( \beta_j \neq 0 \). Then \( B \) is also linearly dependent since applying \( T^{-1} \) to the above eqn gives the non-trivial linear relation \( \sum_i \beta_i v_i = 0 \). As in 2), the converse is proved by applying the forward implication to \( T^{-1} \) & \( T(B) \).
Example on finding a basis for a span

**E.g.** Use the co-ordinate system $C = (1 \ x \ x^2) : \mathbb{F}^3 \rightarrow \mathbb{F}[x]_{\leq 2}$ to determine a basis and hence co-ord system for $W = \text{Span}(1 + x + x^2, -1 + x - 2x^2, 2 + 4x + x^2)$. (You can do this question directly too without co-ordinates).

A Prop 2) shows that $C^{-1}(W) = \text{Span}$

From first year we know how to reduce this spanning set for $C^{-1}(W)$ to a basis.
Matrix representing a linear map wrt co-ordinate systems

Let $T : V \rightarrow W$ be a linear map and $C_V : \mathbb{F}^n \rightarrow V$, $C_W : \mathbb{F}^m \rightarrow W$ be co-ordinate systems. Consider the composite map

$$C_W^{-1} \circ T \circ C_V : \mathbb{F}^n \xrightarrow{C_V} V \xrightarrow{T} W \xrightarrow{C_W^{-1}} \mathbb{F}^m.$$ 

This is a linear map from $\mathbb{F}^n \rightarrow \mathbb{F}^m$ so can be represented by an $m \times n$-matrix in $M_{mn}(\mathbb{F})$.

**Defn**

The *matrix representing $T$ wrt co-ordinate systems $C_V$, $C_W$* is the matrix giving the linear map $C_W^{-1} \circ T \circ C_V$.

**Rem** Recall the co-ordinate systems are given by row matrices $C_V = (v_1 \ldots v_n) \in V^n$, $C_W = (w_1 \ldots w_m) \in W^m$ which are essentially “ordered bases”. In the literature, it is more common to speak of matrices representing $T$ wrt ordered bases.

**Rem** Knowing the representing matrix $A = C_W^{-1} \circ T \circ C_V$ & the co-ordinate systems $C_V$, $C_W$ allows you to recover all the information about $T$ for $T = C_W \circ A \circ C_V^{-1}$. In particular, $A$ will allow us to compute whatever we like about $T$. 

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Lecture 15: Using co-ords to compute kernel, image etc
Semester 2 2013
Let $X$ be a set and consider the $\mathbb{F}$-space of functions $V = \text{Fun}(X, \mathbb{F})$.

**Lemma**

Fix a function $p(x) \in \text{Fun}(X, \mathbb{F})$. The map $p(x) : \text{Fun}(X, \mathbb{F}) \to \text{Fun}(X, \mathbb{F}) : f(x) \mapsto p(x)f(x)$ is linear.

**Proof.** For $f(x), g(x) \in \text{Fun}(X, \mathbb{F})$, $\beta \in \mathbb{F}$, the distributive law

$$p(x)(f(x) + g(x)) = p(x)f(x) + p(x)g(x)$$

shows that the map $p(x)$ respects addition whilst the commutative & associative law

$$p(x)\beta f(x) = \beta p(x)f(x)$$

shows the map $p(x)$ respects scalar multiplication.

**Rem** The notation for this map is bad but common.
Example of finding representing matrix

**E.g.** Consider the linear map $T : \mathbb{R}[x]_{\leq 2} \rightarrow \mathbb{R}[x]_{\leq 3}$ defined by $(Tf)(x) = (x - 1)f'(x) - 2f(x)$ & the co-ordinate systems $C_1 = (1 \times x^2) : \mathbb{R}^3 \rightarrow \mathbb{R}[x]_{\leq 2}$, $C_2 : (1 \times x^2 \times x^3) : \mathbb{R}^4 \rightarrow \mathbb{R}[x]_{\leq 3}$. Find the matrix representing $T$ wrt $C_1, C_2$.

**Rem** Note that $T$ is linear since $T = (x - 1) \circ \frac{d}{dx} - 2 \text{id}$ & we know $x - 1, \frac{d}{dx}, \text{id}$ are linear as are composites & linear combns.

**A** We consider $\mathbb{R}^3 \xrightarrow{C_1} \mathbb{R}[x]_{\leq 2} \xrightarrow{T} \mathbb{R}[x]_{\leq 3} \xrightarrow{C_2^{-1}} \mathbb{R}^4$.

We know from lecture 12, the representing matrix $A \in M_{43}(\mathbb{R})$ has $i$-th column $C_2^{-1} \circ T \circ C_1 e_i$.

1st column is
Isomorphisms preserve kernels & images

The following allows us to compute kernels & images of linear maps from the representing matrix.

**Prop 2**

Let \( T : V \rightarrow W \) be a linear map & \( C_1 : V' \rightarrow V, C_2 : W' \rightarrow W \) be isomorphisms. Let \( T' = C_2^{-1} \circ T \circ C_1 : V' \rightarrow W' \).

1. \( \text{im } T = C_2(\text{im } T') \).
2. \( \ker T = C_1(\ker T') \).

**Proof.** Note that \( C_2 \circ T' = T \circ C_1 \).

For 1), observe that since \( C_1 \) is onto we have \( C_1(V') = V \). Thus

\[
\text{im } T = T(V) = T(C_1(V')) = C_2(T'(V')) = C_2(\text{im } T').
\]

For 2) we first show that \( C_1(\ker T') \subseteq \ker T \) so let \( v' \in \ker T' \). Then
\[
T(C_1v') = C_2(T'v') = C_10 = 0
\]
so \( C_1v' \in \ker T \) & we must have \( C_1(\ker T') \subseteq \ker T \). The reverse inclusion is proved similarly or by applying the inclusion already proved to \( C_1^{-1}, C_2^{-1} \) & \( T', T \) with roles reversed.
Example computing bases for kernels & images

Recall e.g. Consider the linear map $T : \mathbb{R}[x]_{\leq 2} \rightarrow \mathbb{R}[x]_{\leq 3}$ defined by $(Tf)(x) = (x - 1)f'(x) - 2f(x)$ & the co-ordinate systems $C_1 = (1 \times x^2) : \mathbb{R}^3 \rightarrow \mathbb{R}[x]_{\leq 2}$, $C_2 : (1 \times x^2 \ x^3) : \mathbb{R}^4 \rightarrow \mathbb{R}[x]_{\leq 3}$. Compute ker $T$, im $T$ by finding bases for them.

A Recall the representing matrix $C_2^{-1} \circ T \circ C_1$ is

$$A = \begin{pmatrix} -2 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

A basis for ker $A$ is

so by prop 1 & 2, a basis for ker $T$ is

A basis for im $A$ is

so by prop 1 & 2, a basis for im $T$ is