Aim lecture: We relate the important notions of the span of a set of vectors and the image of a linear map. They can be used to give “parametric forms” for vector spaces & in particular, help construct co-ordinate systems.

Prop-Defn

Let $T: V \to W$ be a linear map.

1. For any $V' \leq V$ we have $T(V') = \{ Tv' \in W | v' \in V' \}$ is a subspace of $W$.
2. In particular, the image of $T$, defined to be $\text{im } T = T(V)$ is a subspace of $W$.

Proof. We just need to check closure axioms. Note $0_W = T0_V \in T(V')$ so $T(V')$ is non-empty. Also, for $v', v'' \in V', \beta \in \mathbb{F}$ we have

$$\beta T v' + Tv'' = T(\beta v' + v'') \in T(V')$$

so closure axioms hold & propn-defn is proved.

1. Note that surjectivity of $T$ just means that $\text{im } T = W$.
2. $T(V')$ is an example of a set defined by “parametric form” where the parameter is $v' \in V'$. 
The image of a linear map \( T : \mathbb{F}^n \rightarrow V \)

**Prop-Defn**

Let \( C : \mathbb{F}^n \rightarrow V \) be a linear map given by the row matrix \((v_1 \ldots v_n) \in V^n\). Then

\[
\text{im } T = \mathbb{F}v_1 + \ldots + \mathbb{F}v_n.
\]

We define the \( \mathbb{F}\)-span of \( v_1, \ldots, v_n \) to be the subspace \( \text{im } T \) of \( V \) and denote it \( \text{Span}(v_1, \ldots, v_n) \). In other words, the span is the set of all linear combinations of \( v_1, \ldots, v_n \).

**Proof.** Just calculate

Note that the span does not depend on the order of the vectors so it makes sense to make the following

**Defn**

We say that \( \{v_1, \ldots, v_n\} \) is a *spanning set* for \( V \) or *spans* \( V \) if \( \text{Span}(v_1, \ldots, v_n) = V \) or equiv, the map \( C : \mathbb{F}^n \rightarrow V \) above is surjective. In this case we say \( V \) is *finitely spanned.*
First year example

Make sure you remember from MATH1241/1251, how to do the following question.

E.g. Does \( S = \{1 + 2x^2, 1 - 2x, 2 + x + 5x^2\} \) span \( \mathbb{R}[x]_{\leq 2} \)?

A Since \( S \subseteq \mathbb{R}[x]_{\leq 2} \), need only check given any \( p(x) = a + bx + cx^2 \in \mathbb{R}[x]_{\leq 2} \), is it true that \( p(x) \in \text{Span}(S) \), i.e. can we always solve

\[
\alpha(1 + 2x^2) + \beta(1 - 2x) + \gamma(2 + x + 5x^2) = a + bx + cx^2.
\]
E.g. Let $V$ be the set of solns (in $C^\infty(\mathbb{R})$) to the DE $\frac{d^2y}{dx^2} + y = 0$. Show that $V$ is a subspace of $C^\infty(\mathbb{R})$ by showing it is the span of some set of vectors. Also, find a co-ordinate system for $V$. 
We can generalise the old defn of span to (possibly) infinite sets.

**Defn**
Let $V$ be a vector space and $S \subseteq V$. The *span* of $S$, denoted $\text{Span}(S)$, is defined to be the set of all linear combns of elements of $S$, i.e. set of all vectors of the form $\beta_1v_1 + \ldots + \beta_nv_n$ where $v_1, \ldots, v_n \in S$.

**Prop**
With above notn, $\text{Span}(S)$ is the unique smallest subspace of $V$ containing $S$. More precisely, any subspace $W \leq V$ which contains $S$ must also contain $\text{Span}(S)$.

**Proof.** One can check closure axioms to see $\text{Span}(S)$ in this more general setting is still a subspace. It clearly contains $S$. Suppose $W \leq V$ contains $S$. It closed under linear combns so in particular, contains all linear combns of elts of $S$, i.e. it contains $\text{Span}(S)$. 
Span as an “increasing” function of $S$

We look at the question of how $\text{Span}(S)$ changes as you vary $S$.

**Prop**

Let $V = \mathbb{F}$-space and $S \subseteq V, v \in V$.

1. $\text{Span}(S) \subseteq \text{Span}(S \cup \{v\})$.
2. Equality in 1) holds iff $v \in \text{Span}(S)$.

**Proof.** Note that $\text{Span}(S \cup \{v\})$ is a subspace containing $S$ so the minimality property of $\text{Span}(S)$ ensures 1) holds.

Suppose now equality holds in 1). Then

$$v \in \text{Span}(S \cup \{v\}) = \text{Span}(S).$$

Conversely suppose that $v \in \text{Span}(S)$. Then $\text{Span}(S)$ is a subspace containing $S \cup \{v\}$ so must contain $\text{Span}(S \cup \{v\})$. Hence equality holds in 1).
Let start with a non-zero $v_1 \in \mathbb{R}^3$ and let $S = \{v_1\}$. Then $\text{Span}(S) = \mathbb{R}v_1$ is a line.

Now let’s add $v_2 \in \mathbb{R}^3$ to $S$ so $S = \{v_1, v_2\}$. There are 2 cases.
Defn

We say that a spanning set $S$ for an $F$-space $V$ is *minimal* if any proper subset $S' \subset S$ does not span $V$. Equivalently, by the previous propn, $S$ is a minimal spanning set if each $v \in S$ is not contained in $\text{Span}(S - \{v\})$ i.e. $v$ is not a linear combn of other vectors in $S$.

Any finitely spanned vector space has a minimal spanning set. Indeed, start with any finite spanning set $S$. Then either it is minimal or we can find some $v \in S$ which is a linear combn of other vectors in $S$. We throw away $v$ and repeat until a minimal one is found. This algorithm terminates because $S$ is finite.

All vector spaces have minimal spanning sets, but the proof in general requires Zorn’s lemma so will be omitted.
Example on finding minimal spanning sets

E.g. Find a minimal spanning set for $\text{Span}(S)$ where
$S = \{ p_1(x) = 3 + x, p_2(x) = x + x^2, p_3(x) = 3 + 2x + x^2 \}$.