Throughout, as in lectures, \( \mathbb{F} \) always denotes a field.

(1) [6 marks] For each of the following, say whether the statement is true or false and give a brief reason. You will get one mark for a correct true/false answer, and if your true/false answer is correct then you will get one mark for a good reason.

(i) Let \( V \) be a vector space complement to \( \mathbb{R}(1-x+x^2) \) in \( \mathbb{R}[x]_{\leq 3} \). Then \( \dim V = 2 \).

(ii) Let \( A \in M_{33}(\mathbb{R}) \) be a matrix with

\[
\ker(A - I) = \mathbb{R}(1, 0, 2)^T + \mathbb{R}(3, 1, 1)^T, \quad \ker(A - 2I) = \mathbb{R}(4, 1, 1)^T.
\]

Then \( A \) is diagonalisable.

(iii) Let

\[
A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \in M_{22}(\mathbb{R}).
\]

Then there are infinitely many \( A \)-invariant subspaces of \( \mathbb{R}^2 \).

(2) [4 marks]

Consider the discrete time system \( \mathbf{v}(k + 1) = A\mathbf{v}(k) \) where \( A \in M_{33}(\mathbb{R}) \) has Jordan canonical form \( C^{-1}AC = J = J_2(3) \oplus J_1(1) \). Solve for \( \mathbf{v}(k) \) if

\[
C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{v}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.
\]

(3) [2 marks]

Consider the co-ordinate systems \( C_1 = (1 \ x) : \mathbb{R}^2 \to \mathbb{R}[x]_{\leq 1} \), \( C_2 = (1 \ x \ x^2) : \mathbb{R}^3 \to \mathbb{R}[x]_{\leq 2} \). Let \( T : \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 1} \) be the linear map given by the matrix

\[
\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}
\]

with respect to the co-ordinate systems \( C_2, C_1 \). Find a basis for \( \ker T \).
Consider the following matrix

\[ A = \begin{pmatrix} 3 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}. \]

(i) Find the eigenvalues of \( A \).

(ii) Write down the algebraic and geometric multiplicities of the eigenvalues.

(iii) Find a Jordan canonical form for \( A \).

(iv) Write down a change of co-ordinates matrix \( C \) such that \( J = C^{-1}AC \).

Let \( T, S : V \to V \) be linear maps. Let \( W \leq V \) be a subspace which is both \( T \)-invariant and \( S \)-invariant. Prove that \( W \) is \( S \circ T \)-invariant. Make sure your answer is set out logically and your reasoning is complete.
Throughout, as in lectures, $\mathbb{F}$ always denotes a field.

(1) [6 marks] For each of the following, say whether the statement is true or false and give a brief reason. You will get one mark for a correct true/false answer, and if your true/false answer is correct then you will get one mark for a good reason.

(i) Let $T : M_{23}(\mathbb{F}) \rightarrow M_{22}(\mathbb{F})$ be a surjective linear map. Then the nullity of $T$ is 2.

(ii) The following matrices are similar

$$
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 \\
3 & 5
\end{pmatrix}.
$$

(iii) Let $A \in M_{33}(\mathbb{R})$ be a matrix with

$$
\ker(A - I) = \mathbb{R}(1, 0, 2)^T + \mathbb{R}(8, 0, 1)^T, \ker(A - 2I) = \mathbb{R}(4, 1, 1)^T.
$$

Then $A$ is diagonalisable.

(2) [2 marks]

Consider the co-ordinate systems $C_1 = (1 x) : \mathbb{R}^2 \rightarrow \mathbb{R}[x]_{\leq 1}$, $C_2 = (1 x x^2) : \mathbb{R}^3 \rightarrow \mathbb{R}[x]_{\leq 2}$. Let $T : \mathbb{R}[x]_{\leq 2} \rightarrow \mathbb{R}[x]_{\leq 1}$ be the linear map given by the matrix

$$
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 3
\end{pmatrix}
$$

with respect to the co-ordinate systems $C_2, C_1$. Find a basis for $\ker T$.

(3) [4 marks] Solve the initial value problem $\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t), \quad \mathbf{y}(0) = (1, 0, 1)^T$ where $A \in M_{33}(\mathbb{R})$ has Jordan canonical form $C^{-1}AC = J_2(3) \oplus J_1(1)$ and

$$
C = \begin{pmatrix}
1 & 1 & 0 \\
0 & -1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
$$
Consider the matrix 

\[ A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \end{pmatrix}. \]

(i) Find a basis for the eigenspace \( E_3 \) corresponding to the eigenvalue 3.
(ii) Show that the generalised eigenspace \( E_3(\infty) = \mathbb{C}^3 \).
(iii) Write down a Jordan canonical form \( J \) for \( A \).
(iv) Write down a change of co-ordinates matrix \( C \) such that \( J = C^{-1}AC \).

Let \( T : V \rightarrow V \) be a linear map and \( W, W' \leq V \) be \( T \)-invariant subspaces. Prove that \( W \cap W' \) is also \( T \)-invariant. Make sure your answer is set out logically and your reasoning is complete.