

MATH5725: Galois Theory (2009,S2)
Problem Set 6: Solvability by Radicals.
Finite & Cyclotomic Fields ¹

1. Find the Galois groups of the following polynomials over \mathbb{Q} : i) $x^2 - 3$
 ii) $x^4 - 2$ iii) $x^3 - 2$ iv) $x^5 - 4x + 2$ v) $x^4 - 8x^2 + 9$.
2. Using the definition or otherwise, show that $x^{12} + 6x^9 + 9x^6 - 3x^3 + 6$ is solvable by radicals.
3. Is $x^5 - 4x + 2 \in \mathbb{Q}[x]$ solvable by radicals?
4. Recall the following theorem (proof in Boris's notes on web): A finite field extension K/\mathbb{Q} embeds in a radical extension if and only if its Galois group is solvable. Use this to show any quartic polynomial over \mathbb{Q} is solvable. Show also that any Galois extension K/\mathbb{Q} of degree 27 embeds in a radical extension.
5. Find all the primitive elements for the following field extensions: i) $\mathbb{Q}(\sqrt{2}, \sqrt{5})/\mathbb{Q}$ ii) $\mathbb{Q}(e^{2\pi i/3}, \sqrt[3]{5})/\mathbb{Q}$.
6. Write down the lattice (i.e. set partially ordered by inclusion) of intermediate fields of $\mathbb{F}_{3^{24}}$.
7. What is the Galois group of $\mathbb{F}_{1024}/\mathbb{F}_4$?
8. What is the smallest subfield of $\overline{\mathbb{F}_5}$ containing both \mathbb{F}_{25} and \mathbb{F}_{125} ?
9. Show that $\mathbb{Q}(i\sqrt{3})$ is the cyclotomic field of m -th roots of unity for some m .
10. Let $\zeta_m = e^{2\pi i/m}$. Show that $\mathbb{Q}(\zeta_{12})/\mathbb{Q}(\zeta_3)$ is Galois and compute its Galois group. Compute all the intermediate fields of $\mathbb{Q}(\zeta_{12})$ and hence $\mathbb{Q}(\zeta_3) \cap \mathbb{Q}(\zeta_4)$.
11. Can you construct using a ruler and compass a regular n -gon where:
 i) $n = 15$, ii) $n = 60$, iii) $n = 25$?

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