MATH5725: Galois Theory (2009,S2)
Problem Set 5: Radical extensions & Solvability

1. What are the Galois closures of the following field extensions? i) \( \mathbb{Q}(\sqrt{2})/\mathbb{Q} \) ii) \( \mathbb{Q}(\sqrt[3]{5})/\mathbb{Q} \) iii) \( \mathbb{Q}(\sqrt{2} + \sqrt{2})/\mathbb{Q} \) iv) \( \mathbb{Q}(\sqrt{3} + \sqrt{2})/\mathbb{Q} \).

2. Which of the following field extensions are radical? i) \( \mathbb{Q}(\sqrt[3]{2})/\mathbb{Q} \) ii) \( \mathbb{Q}(\sqrt{2}, \sqrt{7})/\mathbb{Q} \) iii) \( \mathbb{Q}(\sqrt{3} - \sqrt{7})/\mathbb{Q} \) iv) \( \mathbb{Q}(\sqrt{2} + \sqrt{2})/\mathbb{Q} \) v) \( \mathbb{F}_4/\mathbb{F}_2 \) vi) \( \mathbb{C}/\mathbb{R} \).

3. For the radical extensions in the previous question, write down a radical tower and the corresponding normal chain of subgroups with factors cyclic of prime order.

4. Show that any dihedral group is solvable. Compute the derived series of a dihedral group.

5. Show that the alternating group \( A_4 \) is solvable.

6. Find the Sylow subgroups of \( A_4 \).

7. Let \( G \) be a group of order 88, with a normal subgroup of order 11. Show that \( G \) is solvable.

8. Show that \( S_4 \) has a subgroup

\[
D = \{1, (12), (34), (13)(24), (12)(34), (14)(23), (1324), (1423)\}
\]

which is isomorphic to the dihedral group of order 8.

9. Let \( c, d \in \mathbb{Q} \) and \( K = \mathbb{Q}(\sqrt{c + \sqrt{d}}, \sqrt{c - \sqrt{d}}) \). Suppose that \( f(x) = x^4 - 2cx^2 + c^2 - d \) is irreducible and order the roots

\[
\sqrt{c + \sqrt{d}}, -\sqrt{c + \sqrt{d}}, \sqrt{c - \sqrt{d}}, -\sqrt{c - \sqrt{d}}.
\]

Show that \( \text{Gal}K/\mathbb{Q} \) is a subgroup of the group \( D \) in the previous question. Show i) it is the cyclic group \( \langle (1324) \rangle \) if \( d(c^2 - d) \) is a square in \( \mathbb{Q} \), ii) it is \( \{1, (13)(24), (12)(34), (14)(23)\} \) if \( c^2 - d \) is a square in \( \mathbb{Q} \) and finally i) it is \( D \) otherwise.

\footnote{by Daniel Chan}