School of Mathematics University of New South Wales

MATH5725: Galois Theory (2009,S2)
Problem Set 5: Radical extensions & Solvability [1]

1. What are the Galois closures of the following field extensions? i) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ ii) $\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q}$ iii) $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ iv) $\mathbb{Q}(\sqrt{3+\sqrt{2}})/\mathbb{Q}$.

2. Which of the following field extensions are radical? i) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ ii) $\mathbb{Q}(\sqrt{2},\sqrt{7})/\mathbb{Q}$ iii) $\mathbb{Q}(\sqrt{3}-\sqrt{7})/\mathbb{Q}$ iv) $\mathbb{Q}(\sqrt[3]{2+\sqrt{2}})/\mathbb{Q}$ v) $\mathbb{F}_4/\mathbb{F}_2$ vi) $\mathbb{C}/\mathbb{R}$.

3. For the radical extensions in the previous question, write down a radical tower and the corresponding normal chain of subgroups with factors cyclic of prime order.

4. Show that any dihedral group is solvable. Compute the derived series of a dihedral group.

5. Show that the alternating group A_4 is solvable.

6. Find the Sylow subgroups of A_4.

7. Let G be a group of order 88, with a normal subgroup of order 11. Show that G is solvable.

8. Show that S_4 has a subgroup

$$D = \{1, (12), (34), (13)(24), (12)(34), (14)(23), (1324), (1423)\}$$

which is isomorphic to the dihedral group of order 8.

9. Let $c, d \in \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{c+\sqrt{d}}, \sqrt{c-\sqrt{d}})$. Suppose that $f(x) = x^4 - 2cx^2 + c^2 - d$ is irreducible and order the roots

$$\sqrt{c+\sqrt{d}}, -\sqrt{c+\sqrt{d}}, \sqrt{c-\sqrt{d}}, -\sqrt{c-\sqrt{d}}.$$

Show that Gal$K/\mathbb{Q}$ is a subgroup of the group D in the previous question. Show i) it is the cyclic group $\langle(1324)\rangle$ if $d(c^2-d)$ is a square in $\mathbb{Q}$, ii) it is $\{1, (13)(24), (12)(34), (14)(23)\}$ if c^2-d is a square in $\mathbb{Q}$ and finally i) it is D otherwise.

[1] by Daniel Chan