MATH5725: Galois Theory (2009,S2)
Problem Set 4: Galois extensions & correspondence

1. Which of the following field extensions is (finite) Galois?  
   i) \( \mathbb{Q}(\sqrt[3]{7})/\mathbb{Q} \)  
   ii) \( \mathbb{Q}(\sqrt{6})/\mathbb{Q} \)  
   iii) \( \mathbb{Q}(\sqrt[3]{7}, e^{2\pi i/3})/\mathbb{Q} \)  
   iv) \( \mathbb{Q}(\sqrt[3]{7}, e^{2\pi i/3})/\mathbb{Q}(e^{2\pi i/3}) \)  
   v) for prime \( p \), \( \mathbb{F}_p(\sqrt[3]{t})/\mathbb{F}_t \)  
   vi) \( \mathbb{F}_9/\mathbb{F}_3 \)  
   vii) \( \mathbb{C}/\mathbb{R} \)  
   viii) \( \mathbb{Q}(\sqrt{2}, \sqrt[3]{7})/\mathbb{Q}(\sqrt{2}) \)  
   ix) \( \mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}(\sqrt{2}) \)  
   x) \( \mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}(i) \)

2. What are the Galois groups of the field extensions in Q1?

3. Use the Galois correspondence to write down the “lattice” of all the intermediate fields of i) \( \mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q} \) and ii) \( \mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}(i) \).

4. Use the previous question to help you. Consider the field extension \( K/F = \mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}(i) \). What are the following intermediate fields?  
   i) \( \mathbb{Q}(i, 3\sqrt{2} - 2 + 5i) \)  
   ii) \( \mathbb{Q}(i, \sqrt{2} + i\sqrt{2} + 3\sqrt{8}) \)

5. Let \( \omega = e^{2\pi i/3} \) and \( K/F = \mathbb{Q}(\sqrt{2}, \omega, \sqrt{5})/\mathbb{Q}(\omega) \). Compute the Galois group \( \text{Gal} K/F \) and the Galois correspondence. Hint: this example is similar to biquadratic extensions.

6. Shoe that \( K := \mathbb{Q}(\sqrt{2}, i) \) is the splitting field for \( f(x) = x^4 - 2 \) over \( \mathbb{Q} \). Compute the Galois group of \( K/\mathbb{Q}(i) \). Compute the Galois group of \( K/\mathbb{Q} \). Write out the Galois correspondence and use it to determine the intermediate fields \( L \) with \( L/\mathbb{Q} \) Galois. Hint: it may help to plot the roots of \( f(x) \) on the Argand diagram and consider elements of the Galois group as permutations of the corners of the resulting quadrilateral.

7. Complete the example of lecture 9. Let \( \sigma, \tau \) be automorphisms of the field \( K = \mathbb{C}(x, y) \) defined by \( (\sigma f)(x, y) = f(-x, y), (\tau f)(x, y) = f(x, -y) \). Show these are indeed automorphisms and compute all the intermediate fields of \( K/K^{G} \) where \( G = \langle \sigma, \tau \rangle \).