

**MATH5725: Galois Theory (2009,S2)**  
**Problem Set 3: Constructing Field Automorphisms.**  
**Normal & Separable Extensions**<sup>1</sup>

1. Let  $\omega$  be a primitive 5-th root of unity. Show that  $F = \mathbb{Q}(\sqrt[5]{3}, \omega)$  is a splitting field for  $f(x) = x^5 - 3$  over  $\mathbb{Q}$ . i) Show there exists a field automorphism  $\sigma$  of  $\mathbb{Q}(\sqrt[5]{3}, \omega)/\mathbb{Q}$  which sends  $\sqrt[5]{3} \mapsto \sqrt[5]{3}\omega$ . ii) Show there exists a field automorphism  $\sigma$  of  $\mathbb{Q}(\sqrt[5]{3}, \omega)/\mathbb{Q}$  which sends  $\sqrt[5]{3}\omega \mapsto \sqrt[5]{3}\omega^{-1}$  but fixes  $\sqrt[5]{3}$  (hint: look at the field extension  $\mathbb{Q}(\sqrt[5]{3}, \omega)/\mathbb{Q}(\sqrt[5]{3})$ ).
2. Show that there exists a field homomorphism  $\sigma : \mathbb{Q}(\sqrt[3]{2}) \rightarrow \mathbb{C}$  over  $\mathbb{Q}$  which maps  $\sqrt[3]{2} \mapsto \sqrt[3]{2}e^{2\pi i/3}$ . If  $d$  is the separable degree of  $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$  then what is  $d$  and what are the  $d$  field homomorphisms  $\mathbb{Q}(\sqrt[3]{2}) \rightarrow \overline{\mathbb{Q}}$ ?
3. Show that  $\mathbb{Q}(\sqrt{3}, \sqrt{10})$  is a splitting field for  $p(x) = (x^2 - 3)(x^2 - 10)$  over  $\mathbb{Q}$ . Show that  $\text{Gal}(\mathbb{Q}(\sqrt{3}, \sqrt{10})/\mathbb{Q})$  is isomorphic to a subgroup  $G$  of the permutation group  $\text{Perm } S$  where  $S = \{\sqrt{3}, -\sqrt{3}, \sqrt{10}, -\sqrt{10}\}$ . Use the uniqueness of splitting fields to find  $G$ . Hint: Show that  $G$  has at most 4 elements and construct those field automorphisms individually.
4. Find  $\text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{7})/\mathbb{Q})$ .
5. Which of the following are normal field extensions i)  $\mathbb{Q}(\sqrt{6})/\mathbb{Q}$ , ii)  $\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q}$ , iii)  $\mathbb{Q}(\sqrt[3]{5}, e^{2\pi i/3})/\mathbb{Q}$ , iv)  $\mathbb{Q}(\sqrt[3]{5}, e^{2\pi i/3})/\mathbb{Q}(\sqrt[3]{5})$ , v)  $\mathbb{Q}(\sqrt[3]{5}, e^{2\pi i/3})/\mathbb{Q}(e^{2\pi i/3})$ , vi)  $\mathbb{F}_p(\sqrt[p]{t})/\mathbb{F}_p(t)$ , vii) a degree 2 field extension  $K/F$  if the characteristic of  $F$  is not 2, viii)  $\mathbb{Q}(\sqrt[6]{2})/\mathbb{Q}(\sqrt{2})$ ?
6. Which of the following extensions are separable? i)  $\mathbb{C}/\mathbb{R}$ , ii)  $\mathbb{Q}(\sqrt[8]{11})/\mathbb{Q}$ , iii)  $\mathbb{F}_3(\sqrt{2})/\mathbb{F}_3$ , iv) the splitting field for  $x^2 + x + 1$  over  $\mathbb{F}_3$ , v) the splitting field for  $x^9 + 2t$  over  $\mathbb{F}_3(t)$ .

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