

MATH5725: Galois Theory (2009,S2)
Problem Set 2: Splitting fields¹

Below, F will always denote a field.

1. Write down a splitting field for $x^3 - 1$ over \mathbb{Q} in the form i) $\mathbb{Q}(\alpha)$ for some element $\alpha \in \mathbb{C}$ and ii) $\mathbb{Q}[x]/\langle p(x) \rangle$ for some polynomial $\mathbb{Q}[x]$.
2. Let $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ be the field with 3 elements. Write down a splitting field K for $x^3 - 1$ over \mathbb{F}_3 . How many elements does K have?
3. Write down a splitting field for $x^3 - 5$ over i) \mathbb{Q} ii) $\mathbb{Q}(\sqrt[3]{5})$ iii) \mathbb{R} .
4. If $f(x) \in F[x]$ is a linear polynomial, what is the splitting field for $f(x)$ over F ?
5. What is the splitting field for $x^3 - 6x$ over \mathbb{R} ?
6. Do the exercise in lecture 2. Let $F \subseteq L \subseteq K$ be a tower of field extensions and $f(x) \in F[x]$. i) Show that if K is a splitting field for $f(x)$ over F , then it is also a splitting field for $f(x)$ over L . ii) Show the converse holds if L is generated over F by some of the roots of $f(x)$.
7. Repeat the previous question but for a family of polynomials $\{f_i(x)\} \subseteq F[x] - F$.
8. What is a splitting field for $\{x^2 - 2, x^2 - 3, x^2 - 5\}$ over i) \mathbb{Q} ii) $\mathbb{Q}(\sqrt{2})$ iii) $\mathbb{Q}(\sqrt[3]{7})$?
9. Show that if K is a splitting field for $f(x) \in F[x]$ over F , then it contains all the roots of $f(x)$ in any extension field L of K .
10. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \dots)$. i) For any positive rational number q , show that $\sqrt{q} \in F$. ii) Show that any subfield L of F which is finite over \mathbb{Q} has degree a power of 2. iii) Hence or otherwise show that $\sqrt[3]{5} \notin F$.

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