School of Mathematics University of New South Wales

MATH5725: Galois Theory (2009,S2)
Problem Set 1: revision [1]

The purpose of this problem set is to make sure you are familiar with all the basic notions concerning fields that you learnt in MATH3711. Hopefully you will remember the following concepts: characteristic of a field, algebraic extensions, finite extensions, simple field extensions, degree of a field extension, tower of field extensions, minimal polynomials, algebraically closed.

1. Which of the following rings are fields: $\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{R}[x], \mathbb{R}(x), \mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}[i]/\langle 2+i \rangle, \mathbb{Q}[\sqrt{2}], \mathbb{Q}[\pi], \mathbb{Q}(\pi), M_3(\mathbb{R})$?
2. What are the characteristics of $\mathbb{C}, \mathbb{R}(x), \mathbb{F}_4, \mathbb{F}_{27}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}[i]/\langle 2+i \rangle, \mathbb{Q}[\sqrt{2}]$?
3. Write down a $\mathbb{Q}$-basis for the field $\mathbb{Q}(\sqrt[3]{2})$ and determine the degree $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$ of the field extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$.
4. Is $\alpha := \sqrt{2+\sqrt{5}}$ algebraic over $\mathbb{Q}$? If so, determine its minimal polynomial and hence the degree of $\mathbb{Q}(\alpha)/\mathbb{Q}$.
5. Let E/F be a finite field extension of prime degree. Show that E is a simple extension of F.
6. Is the field extension $\mathbb{Q}(\sqrt{2}, \sqrt[3]{3})/\mathbb{Q}$ finite, and if so, what is its degree? Is it algebraic?
7. Give an example of a field extension which is algebraic but not finite.
8. Suppose $K/L, L/F$ are field extensions of degrees 2 and 3 respectively. Is K/F finite and if so, what is its degree? Is K/F algebraic?
9. What is the degree of $\mathbb{Q}(i, \sqrt[4]{3})/\mathbb{Q}$? Write down a $\mathbb{Q}$-basis for $\mathbb{Q}(i, \sqrt[4]{3})$.

[1] by Daniel Chan