1. (36 marks total) Justify your answers with a brief explanation (but be careful to mention the key points).

   a) Let $K/F$ be a Galois extension of degree $n$. What is the order of the Galois group $\text{Gal } K/F$?
   
   b) What is $\text{Gal } \mathbb{F}_{64}/\mathbb{F}_2$?
   
   c) Is the Galois group of $x^6 - 3x^3 - 6$ over $\mathbb{Q}$ isomorphic to $S_6$?
   
   d) Is the dihedral group of order 10 solvable?
   
   e) What is the Galois closure of $\mathbb{Q}(\sqrt[5]{2})/\mathbb{Q}$?
   
   f) Let $L/K, K/F$ be Galois field extensions.
      
      i) Is $L/F$ separable?
      
      ii) Is $L/F$ Galois?
   
   g) Is $x^5 - 4x + 2 \in \mathbb{Q}[x]$ solvable by radicals?
   
   h) Give an example of a Galois extension $K$ of $\mathbb{Q}(i)$ whose Galois group is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
      
      i) For $K = \mathbb{Q}(\sqrt[5]{2}, \sqrt[3]{3})$, compute $N_{K/\mathbb{Q}}(2 + \sqrt[5]{2})$.
   
   j) Let $K/F$ be a finite abelian extension and $L$ be an intermediate field. Is $L/F$ a Galois extension?
   
   k) Let $L/K, K/F$ be purely inseparable field extensions. Is $L/F$ purely inseparable?

2. (8 marks total) Let $K = \mathbb{Q}(e^{2\pi i/3}, \sqrt[5]{5})$ and consider the field extension $K/\mathbb{Q}$.

   a) (3 marks) What is the Galois group $G$ of $K/\mathbb{Q}$? Justify your answer.
   
   b) (5 marks) Write down all the intermediate fields of $K/\mathbb{Q}$ and the corresponding subgroups of $G$. You need not explain your computations.

3. (8 marks) Let $K/F$ be a finite Galois extension with Galois group $G \simeq G_1 \times G_2$ for some groups $G_1, G_2$. Show that there exist intermediate fields $L_1, L_2$ satisfying the following three conditions.

   a) For $i = 1, 2$, $L_i/F$ is Galois with Galois group isomorphic to $G_i$.
   
   b) $K$ is the smallest field containing both $L_1$ and $L_2$. 

Please see over . . .
c) \( L_1 \cap L_2 = F \).

Make sure you fully justify your answer.

4. (5 marks total) Consider the field automorphism \( \sigma : \mathbb{C}(t) \rightarrow \mathbb{C}(t) \) defined by

\[(\sigma f)(t) := f(t^{-1}).\]

Let \( G \) be the cyclic group generated by \( \sigma \).

a) (1 mark) What is the order of \( G \)?

b) (2 marks) What is \([\mathbb{C}(t) : \mathbb{C}(t + t^{-1})]\)?

c) (2 marks) Prove that \( \mathbb{C}(t)^G = \mathbb{C}(t + t^{-1}) \).

5. (6 marks total) This question concerns the existence of the so called “maximal abelian subextension”.

a) (3 marks) Let \( G \) be a (not necessarily Hausdorff) topological group and \( N \) be a normal subgroup. Prove that the closure of \( N \) is also a normal subgroup.

b) (3 marks) Let \( K/F \) be a Galois extension. Show that there exists an intermediate field \( K_{ab} \) satisfying the following two conditions.

i) \( K_{ab}/F \) is an abelian extension.

ii) If \( L \) is any intermediate field of \( K/F \) such that \( L/F \) is also an abelian extension, then \( L \subseteq K_{ab} \).

6. (7 marks total) Recall that we have the following pro-finite groups

\[ \hat{\mathbb{Z}} = \lim_{\rightarrow n} \mathbb{Z}/n\mathbb{Z}, \quad \hat{\mathbb{Z}}_p = \lim_{\rightarrow j} \mathbb{Z}/p^j\mathbb{Z} \]

where \( p \) is any prime.

a) (3 marks) Show that

\[ \hat{\mathbb{Z}} \simeq \prod_{p, \text{ prime}} \hat{\mathbb{Z}}_p. \]

b) (2 marks) Let \( \phi : \hat{\mathbb{Z}} \rightarrow \hat{\mathbb{Z}}_p \) denote the natural projection. Show that \( H := \ker \phi \) is a closed subgroup of \( \hat{\mathbb{Z}} \).

c) (2 marks) Suppose \( q \) is a prime and let us identify the absolute Galois group \( \text{Gal } \overline{\mathbb{F}}_q/\mathbb{F}_q \) with \( \hat{\mathbb{Z}} \) as in lectures so that we may consider \( H \) as a subgroup of \( \text{Gal } \overline{\mathbb{F}}_q/\mathbb{F}_q \). Determine with reason, the fixed field \( \mathbb{F}_q^H \).