MATH1231 Algebra, 2017
Chapter 7: Linear maps

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Chapter overview

$\mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x,y) = 2x - 3y$ is an example of a linear function, $g(x,y) = x^2 - 5y$ is not.

In this chapter, study more generally linear transformations $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$.

Even more generally, study linear $T: V \rightarrow W$ where $V$, $W$ = vector spaces $\mathbb{F}$.

Recall $V$ is the domain of $T$ & $W$ the codomain of $T$.

We'll generalise systems of linear equations $A\mathbf{x} = \mathbf{b}$ to "linear equations" of form $T\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \in W$, $\mathbf{x} \in V$.

Often abbreviate $T(x) = T\mathbf{x}$. 

Daniel Chan (UNSW) 7.1 Introduction to Linear Maps
Chapter overview

- \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) defined by \( f\begin{pmatrix} x \\ y \end{pmatrix} = 2x - 3y \) is example of a linear function, \( g\begin{pmatrix} x \\ y \end{pmatrix} = x^2 - 5y \) is not.
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- \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) defined by \( f(\begin{pmatrix} x \\ y \end{pmatrix}) = 2x - 3y \) is example of a linear function, \( g(\begin{pmatrix} x \\ y \end{pmatrix}) = x^2 - 5y \) is not.

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- In this chapter, study more generally **linear transformations** $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$.
- Even more gen, study linear $T : V \rightarrow W$ where $V, W =$ vector spaces over $\mathbb{F}$.
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- We’ll generalise systems of linear equations $Ax = b$ to “linear equations” of form $Tx = b$ where $b \in W, x \in V$.

Often abbrev $T(x) = Tx$. 
Addition Condition

To define linear map, first consider Addition Condition. We say $T: V \rightarrow W$ satisfies the addition condition, if $T(v + v') = T(v) + T(v')$ for all $v, v' \in V$.

E.g. $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $T(x, y) = 2x - 3y$ satisfies the addn condn since given $(x, y), (x', y') \in \mathbb{R}^2$, $T((x, y) + (x', y')) = T(x, y) + T(x', y')$.

Warning: The + on the two sides of the equation are different!
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Scalar Multiplication Condition

We say $T: V \to W$ satisfies the scalar multiplication condition, if $T(\lambda v) = \lambda T(v)$ for all $\lambda \in F$ and $v \in V$.

E.g. $T: \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = 2x - 3y$ satisfies the scalar multiplication condition since given $(x, y) \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$, $T(\lambda(x, y)) = \lambda T(x, y)$.

Warning: The scalar multiplication on both sides of the equation is different!
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**Example.** $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $T(x, y) = 2x - 3y$ satisfies the scalar multiplication condition since given $(x, y) \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$

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Definition

Let $V$, $W = \text{vector spaces}$.

A function $T: V \to W$ is called a linear map or a linear transformation if following both hold.

Addition Condition.

$T(v + v') = T(v) + T(v') \quad \text{for all } v, v' \in V$.

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E.g. $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $T(x,y) = 2x - 3y$ is linear.
Definition

Let $V, W = \text{vector spaces over } F$. 

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**E.g.** $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $T(\begin{pmatrix} x \\ y \end{pmatrix}) = 2x - 3y$ is linear.
Sample question: showing a function is linear.

**Example**

Show that the function $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$T(x) = \begin{pmatrix} 4x_2 - 3x_3 \\ x_1 + 2x_2 \end{pmatrix} \quad \text{for} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

is a linear map.

**Solution**
Proposition.

If $T : V \rightarrow W$ is a linear map, then $T(0) = 0$. 

Example

Show that the function $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$T(x_1, x_2) = \begin{pmatrix} x_1 + x_2 \\ x_2 - 2x_1 \\ 0 \end{pmatrix}$$

is not linear.
Proposition.

*If* $T : V \rightarrow W$ *is a linear map, then* $T(0) = 0$.

**Proof.** $T(0) = T(00) = 0 \cdot T(0) = 0$. 

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Soln Daniel Chan (UNSW)
Linear maps preserve zero. 

**Proposition.**

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**Example**

Show that the function $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} x_1 + x_2 \\ x_2 - 2 \\ x_1 \end{pmatrix}$ is not linear.

**Soln**
Another non-linear example

Example

Show that the function $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + x_2, x_2^2)$ is not linear.

Solution

Checking $T(0) = 0$ here tells you nothing about linearity.

Suffice to check that $T(\lambda v)$ fails for single choice of pair $\lambda, v$. 
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Checking $T(0) = 0$ here tells you nothing about linearity. Suffice check that $T(\lambda \mathbf{v}) = \lambda T \mathbf{v}$ fails for single choice of pair $\lambda, \mathbf{v}$.
Theorem

The function $T : V \to W$ is a linear map iff for all $\lambda \in F$ and $v_1, v_2 \in V$

$$T(\lambda v_1 + v_2) = \lambda T(v_1) + T(v_2).$$

Remark

This means that a linear map $T$ has the special property that it sends the line $x = a + \lambda v$ to the line $x = T(a) + \lambda T(v)$ or point $T(a)$ if $T(v) = 0$.

E.g. Differentiation is a linear map.

More precisely, we define $T : P \to P$ by $T_p = \frac{dp}{dx}$. Then for $p, q \in P$, $\lambda \in \mathbb{R}$

$$T(\lambda p + q) = \frac{\lambda dp}{dx} + \frac{dq}{dx}.$$
Alternate characterisation of linearity

We can combine the addn condn & scalar multn condn into one!

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Theorem

If $T : V \to W$ is linear map, $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ & $\lambda_1, \ldots, \lambda_n$ are scalars, then

$$T(\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n) = \lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_n T(\mathbf{v}_n).$$
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If $T : V \rightarrow W$ is linear map, $v_1, \ldots, v_n \in V$ & $\lambda_1, \ldots, \lambda_n$ are scalars, then

$$T(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 T(v_1) + \cdots + \lambda_n T(v_n).$$

Example

If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a function such that

$$T \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad T \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$ 

Show that $T$ is not linear.

Solution
Example

Given that $T$ is a linear map and

$$
T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},
T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix},
T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.
$$

Find $T \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Solution
Linear maps are determined by the values on a spanning set

Previous eg illustrates the following general result.

\[ \text{Theorem} \]

\[ T : V \rightarrow W \text{ is linear} \quad \text{and} \quad V = \text{span}(v_1, \ldots, v_m) \]

Then \( T \) is completely determined by the \( m \) values \( T(v_1), \ldots, T(v_m) \).

Compare with the following:

An affine linear function \( f(x) = mx + b \) is determined by two of its values \( f(x_1), f(x_2) \), since its graph is a line which is determined by two points.
Linear maps are determined by the values on a spanning set

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**Theorem**

Let $T : V \rightarrow W$ be linear & $V = \text{span}(v_1, \ldots, v_m)$. Then $T$ is completely determined by the $m$ values $T v_1, \ldots, T v_m$. 
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Let \( T : V \rightarrow W \) be linear & \( V = \text{span}(v_1, \ldots, v_m) \). Then \( T \) is completely determined by the \( m \) values \( Tv_1, \ldots, Tv_m \).

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Compare with the following:

An affine linear function \( f(x) = mx + b \) is determined by two of its values \( f(x_1), f(x_2) \), since
Linear maps are determined by the values on a spanning set

Previous eg illustrates the following general result.

**Theorem**

Let $T : V \rightarrow W$ be linear & $V = \text{span}(v_1, \ldots, v_m)$. Then $T$ is completely determined by the $m$ values $T v_1, \ldots, T v_m$.

Compare with the following:

An affine linear function $f(x) = mx + b$ is determined by two of its values $f(x_1), f(x_2)$, since its graph is a line which is determined by two points.
Theorem

For each $m \times n$ matrix $A$, the function $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$, defined by

$$T_A(x) = Ax \quad \text{for} \quad x \in \mathbb{R}^n,$$

is a linear map called the associated linear map.

Proof.
Example of reflection

Example

Let \( A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), describe the associated linear map \( T_A \) geometrically as a mapping \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \).

Solution
Matrix Representation Theorem

Conversely, given linear $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, we can find an $m \times n$ matrix $A$ such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$.
Matrix Representation Theorem

Conversely, given linear $T : \mathbb{R}^n \to \mathbb{R}^m$, we can find an $m \times n$ matrix $A$ such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$. In this case, we say $A$ is a matrix representing $T$.

Example

Given that $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x y) = \begin{bmatrix} x + 2y \\ 2x - y \\ y \end{bmatrix}$ is linear. Find the matrix $A$ representing $T$.

Solution
Matrix Representation Theorem

Conversely, given linear \( T : \mathbb{R}^n \to \mathbb{R}^m \), we can find an \( m \times n \) matrix \( A \) such that \( T(x) = Ax \) for all \( x \in \mathbb{R}^n \). In this case, we say \( A \) is a matrix representing \( T \).

Example

Given that \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) defined by \( T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x - y \\ y \end{pmatrix} \) is linear. Find the matrix \( A \) representing \( T \).
Matrix Representation Theorem

Conversely, given linear \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \), we can find an \( m \times n \) matrix \( A \) such that \( T(x) = Ax \) for all \( x \in \mathbb{R}^n \). In this case, we say \( A \) is a matrix representing \( T \).

Example

Given that \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) defined by \( T \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + 2y \\ 2x - y \\ y \end{array} \right) \) is linear. Find the matrix \( A \) representing \( T \).

Solution
Theorem

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let the vectors $e_j$ for $1 \leq j \leq n$ be the standard basis vectors for $\mathbb{R}^n$.

Then the $m \times n$ matrix $A = (T e_1 | T e_2 | ... | T e_n)$ has the property that $T(x) = Ax$ for all $x \in \mathbb{R}^n$.

E.g. In the example of the previous slide, $T e_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $T e_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, so the representing matrix is

$$
\begin{pmatrix}
1 & 2 \\
0 & 1 \\
\end{pmatrix}
$$
Theorem

Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear map and let the vectors \( e_j \) for \( 1 \leq j \leq n \) be the standard basis vectors for \( \mathbb{R}^n \).
Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let the vectors $e_j$ for $1 \leq j \leq n$ be the standard basis vectors for $\mathbb{R}^n$. Then the $m \times n$ matrix

$$A = (Te_1 \mid Te_2 \mid \ldots \mid Te_n)$$

has the property that

$$T(x) = Ax \quad \text{for all} \quad x \in \mathbb{R}^n.$$
Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let the vectors $e_j$ for $1 \leq j \leq n$ be the standard basis vectors for $\mathbb{R}^n$. Then the $m \times n$ matrix

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E.g. In the example of the previous slide,
Theorem

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map and let the vectors $e_j$ for $1 \leq j \leq n$ be the standard basis vectors for $\mathbb{R}^n$. Then the $m \times n$ matrix

$$A = (Te_1 | Te_2 | \ldots | Te_n)$$

has the property that

$$T(x) = Ax \text{ for all } x \in \mathbb{R}^n.$$

E.g. In the example of the previous slide,

$Te_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $Te_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ so the representing matrix is
A 5-point star with vertices $A(1, 5), B(4, 3), C(3, -1), D(-1, -1)$ and $E(-2, 3)$.
Example

Find and draw the image of the 5-point star under the linear map $T_M$ defined by the matrix $M = \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$.

Solution
Rotation about 0 is linear

Consider the map $R_\alpha$, which rotates the $\mathbb{R}^2$ plane through an angle $\alpha$ anticlockwise about the origin.
Rotation about $0$ is linear

Consider the map $R_\alpha$, which rotates the $\mathbb{R}^2$ plane through an angle $\alpha$ anticlockwise about the origin.

One can show geometrically that $R_\alpha$ is a linear map see Section 7.3. example 3.
Consider the map \( R_\alpha \), which rotates the \( \mathbb{R}^2 \) plane through an angle \( \alpha \) anticlockwise about the origin.

One can show geometrically that \( R_\alpha \) is a linear map see Section 7.3. example 3.

**Example**

Find the matrix \( A \) representing \( R_\alpha \).
Rotation about 0 is linear

Consider the map $R_\alpha$, which rotates the $\mathbb{R}^2$ plane through an angle $\alpha$ anticlockwise about the origin.

One can show geometrically that $R_\alpha$ is a linear map see Section 7.3. example 3.

Example

Find the matrix $A$ representing $R_\alpha$.

Solution
Projection onto $b$ is linear

Recall given $b \in \mathbb{R}^n$ we have a projection map $\text{proj}_b : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which sends $x \mapsto \text{proj}_b x$.

**Proposition**

$$\text{proj}_b x = \frac{1}{||b||^2} b b^T x$$

Hence $\text{proj}_b$ is linear being the linear map associated to the matrix $A = \frac{1}{||b||^2} b b^T$.

**Proof.**

Note $A x = \frac{1}{||b||^2} b b^T x = \frac{1}{||b||^2} b (b \cdot x) = b \cdot x \frac{1}{||b||^2} b = \text{proj}_b x$ from the formula for $\text{proj}_b$ given in MATH1131 Daniel Chan (UNSW) 7.3 Linear maps from geometry
Projection onto $\mathbf{b}$ is linear

Recall given $\mathbf{b} \in \mathbb{R}^n$ we have a projection map $\text{proj}_b : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which sends $\mathbf{x} \mapsto \text{proj}_b \mathbf{x}$.

**Proposition** \[ \text{proj}_b \mathbf{x} = \frac{1}{|\mathbf{b}|^2} \mathbf{b} (\mathbf{b} \cdot \mathbf{x}) \]

Hence $\text{proj}_b$ is linear being the linear map associated to the matrix $A = \frac{1}{|\mathbf{b}|^2} \mathbf{b} \mathbf{b}^T$.

**Proof.** Note \[ A \mathbf{x} = \frac{1}{|\mathbf{b}|^2} \mathbf{b} (\mathbf{b} \cdot \mathbf{x}) = \mathbf{b} \cdot \mathbf{x} \frac{1}{|\mathbf{b}|^2} \mathbf{b} = \text{proj}_b \mathbf{x} \] from the formula for $\text{proj}_b \mathbf{x}$ given in MATH1131 Daniel Chan (UNSW) 7.3 Linear maps from geometry 24 / 43
Projection onto $b$ is linear

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**Proposition**

$$\text{proj}_b x = \frac{1}{|b|^2} bb^T x$$
Projection onto \( \mathbf{b} \) is linear

Recall given \( \mathbf{b} \in \mathbb{R}^n \) we have a projection map \( \text{proj}_\mathbf{b} : \mathbb{R}^n \longrightarrow \mathbb{R}^n \) which sends \( \mathbf{x} \mapsto \text{proj}_\mathbf{b}\mathbf{x} \).

**Proposition**

\[
\text{proj}_\mathbf{b}\mathbf{x} = \frac{1}{|\mathbf{b}|^2} \mathbf{b} \mathbf{b}^T \mathbf{x}
\]

Hence \( \text{proj}_\mathbf{b} \) is linear being the linear map associated to the matrix
Projection onto $\mathbf{b}$ is linear

Recall given $\mathbf{b} \in \mathbb{R}^n$ we have a projection map $\text{proj}_\mathbf{b} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which sends $\mathbf{x} \mapsto \text{proj}_\mathbf{b} \mathbf{x}$.

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$$\text{proj}_\mathbf{b} \mathbf{x} = \frac{1}{|\mathbf{b}|^2} \mathbf{b} \mathbf{b}^T \mathbf{x}$$

Hence $\text{proj}_\mathbf{b}$ is linear being the linear map associated to the matrix

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Projection onto $b$ is linear

Recall given $b \in \mathbb{R}^n$ we have a projection map $\text{proj}_b : \mathbb{R}^n \to \mathbb{R}^n$ which sends $x \mapsto \text{proj}_b x$.

**Proposition**

$$\text{proj}_b x = \frac{1}{|b|^2} bb^T x$$

Hence $\text{proj}_b$ is linear being the linear map associated to the matrix

$$A = \frac{1}{|b|^2} bb^T.$$ 

**Proof.**

Note

$$Ax = \frac{1}{|b|^2} bb^T x = \frac{1}{|b|^2} b (b \cdot x) = \frac{b \cdot x}{|b|^2} b = \text{proj}_b x$$
Projection onto $\mathbf{b}$ is linear

Recall given $\mathbf{b} \in \mathbb{R}^n$ we have a projection map $\text{proj}_\mathbf{b} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which sends $\mathbf{x} \mapsto \text{proj}_\mathbf{b}\mathbf{x}$.

**Proposition**

$$\text{proj}_\mathbf{b}\mathbf{x} = \frac{1}{|\mathbf{b}|^2} \mathbf{b}\mathbf{b}^T \mathbf{x}$$

Hence $\text{proj}_\mathbf{b}$ is linear being the linear map associated to the matrix

$$A = \frac{1}{|\mathbf{b}|^2} \mathbf{b}\mathbf{b}^T.$$

**Proof.**

Note

$$A\mathbf{x} = \frac{1}{|\mathbf{b}|^2} \mathbf{b}\mathbf{b}^T \mathbf{x} = \frac{1}{|\mathbf{b}|^2} \mathbf{b}(\mathbf{b} \cdot \mathbf{x}) = \frac{\mathbf{b} \cdot \mathbf{x}}{|\mathbf{b}|^2} \mathbf{b} = \text{proj}_\mathbf{b}\mathbf{x}$$

from the formula for $\text{proj}_\mathbf{b}\mathbf{x}$ given in MATH1131.
Sample projection

Let \( b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( T = \text{proj}_b : \mathbb{R}^2 \to \mathbb{R}^2 \).

i) Find the matrix \( A \) representing \( T \).

ii) Check your answer by computing the linear map associated to the matrix \( A \) you found.

Solution

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Sample projection

Example

Let \( b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( T = \text{proj}_b : \mathbb{R}^2 \to \mathbb{R}^2 \).
Sample projection

Example

Let $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T = \text{proj}_\mathbf{b} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

i) Find the matrix $A$ representing $T$. 

Solution

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Sample projection

Example

Let \( b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( T = \text{proj}_b : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \).

i) Find the matrix \( A \) representing \( T \). ii) Check your answer by computing the linear map associated to the matrix \( A \) you found.

Solution
Let $T: V \to W$ be a linear map.

**Proposition-Definition**

The kernel of $T$ (written $\ker(T)$ or $\ker(T)$) is the set,

$\ker(T) = \{ v \in V \mid T(v) = 0 \} \subseteq V$.

Let $A \in M_{mn}(\mathbb{R})$ & $T_A: \mathbb{R}^n \to \mathbb{R}^m$ be the associated linear map.

We define $\ker(A) = \ker(T_A) = \{ v \in \mathbb{R}^n \mid Av = 0 \}$.

$\ker(T)$ is a subspace of $V$.

E.g. Is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in $\ker(2I - I)$?

E.g. Consider the differentiation map $T: \mathbb{P} \to \mathbb{P}$,

$(Tp)(x) = p'(x)$.

$\ker(T) = \{ p \in \mathbb{P} \mid dp/dx = 0 \} = \mathbb{P}_0$, the subspace of all constant polynomials.
Kernels of linear maps

Let $T : V \rightarrow W$ be a linear map.
Kernels of linear maps

Let $T : V \to W$ be a linear map.

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The **kernel** of $T$ (written $\ker(T)$ or $\text{ker } T$) is the set,

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$\ker T$ is a subspace of $V$.

**E.g.** Is $\left(\begin{array}{c} 1 \\ 2 \end{array}\right)$ in $\ker(2 \ -1)$?
Kernels of linear maps

Let $T : V \rightarrow W$ be a linear map.

**Proposition-Definition**

The **kernel** of $T$ (written $\ker(T)$ or $\ker T$) is the set,

$$\ker(T) = \{ v \in V | T(v) = 0 \} \subseteq V.$$

Let $A \in M_{mn} (\mathbb{R}) \& T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\ker A = \ker T_A = \{ v \in \mathbb{R}^n : Av = 0 \}.$$ 

$\ker T$ is a subspace of $V$.

**E.g.** Is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in $\ker(2 - 1)$?

**E.g.** Consider the differentiation map $T : \mathbb{P} \rightarrow \mathbb{P}, (Tp)(x) = p'(x)$. 

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Kernels of linear maps

Let \( T : V \to W \) be a linear map.

**Proposition-Definition**

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\[
\ker(T) = \{ v \in V \mid T(v) = 0 \} \subseteq V.
\]

Let \( A \in M_{mn}(\mathbb{R}) \) & \( T_A : \mathbb{R}^n \to \mathbb{R}^m \) be the assoc linear map. We define \( \ker A = \ker T_A = \{ v \in \mathbb{R}^n : Av = 0 \} \).

\( \ker T \) is a subspace of \( V \).

**E.g.** Is \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) in \( \ker(2 \ -1) \)?

**E.g.** Consider the differentiation map \( T : \mathbb{P} \to \mathbb{P}, (Tp)(x) = p'(x) \).

\( \ker T = \{ p \in \mathbb{P} \mid \frac{dp}{dx} = 0 \} = \mathbb{P}_0 \)
Kernels of linear maps

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Proof that kernels are subspaces

Let $T: V \rightarrow W$ be a linear map. We prove that $\ker T$ is a subspace of $V$ by checking closure axioms.

Proof.

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Proof that kernels are subspaces

Let $T : V \rightarrow W$ be a linear map. We prove that $\ker T$ is a subspace of $V$ by checking closure axioms.
Proof that kernels are subspaces

Let $T : V \rightarrow W$ be a linear map. We prove that $\ker T$ is a subspace of $V$ by checking closure axioms.

Proof.
Let $T: V \rightarrow W$ be a linear map.

**Proposition-Definition**

The image of $T$ is the set of all function values of $T$, that is,

$$\text{im}(T) = \{ w \in W : w = T(v) \text{ for some } v \in V \} \subseteq W.$$

Let $A \in \mathbb{M}_{mn}(\mathbb{R})$ & $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the associated linear map.

We define $\text{im}A = \text{im}T_A = \{ b \in \mathbb{R}^m : b = Ax \text{ for some } x \in \mathbb{R}^n \} = \text{col}(A)$.

$\text{im}T$ is a subspace of $W$.

**Remark**

Proof omitted, but note we already know $\text{col}(A)$ is a subspace as it is the span of the columns of $A$. 

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Images of linear maps

Let $T : V \rightarrow W$ be a linear map.
Images of linear maps

Let $T: V \to W$ be a linear map.

**Proposition-Definition**

The **image** of $T$ is the set of all function values of $T$, that is,

$$\text{im}(T) = \{ w \in W : w = T(v) \text{ for some } v \in V \} \subseteq W.$$
Let $T : V \to W$ be a linear map.

**Proposition-Definition**

The **image** of $T$ is the set of all function values of $T$, that is,

$$\text{im}(T) = \{ w \in W : w = T(v) \text{ for some } v \in V \} \subseteq W.$$ 

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the assoc linear map.
Let $T : V \rightarrow W$ be a linear map.

**Proposition-Definition**

The **image** of $T$ is the set of all function values of $T$, that is,

$$\text{im}(T) = \{w \in W : w = T(v) \text{ for some } v \in V\} \subseteq W.$$ 

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\text{im} A = \text{im} T_A = \{b \in \mathbb{R}^m : b = Ax \text{ for some } x \in \mathbb{R}^n\} = \text{col}(A).$$
Images of linear maps

Let \( T : V \to W \) be a linear map.

**Proposition-Definition**

The **image** of \( T \) is the set of all function values of \( T \), that is,

\[
\text{im}(T) = \{ w \in W : w = T(v) \text{ for some } v \in V \} \subseteq W.
\]

Let \( A \in M_{mn}(\mathbb{R}) \) & \( T_A : \mathbb{R}^n \to \mathbb{R}^m \) be the assoc linear map. We define

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\text{im } A = \text{im } T_A = \{ b \in \mathbb{R}^m : b = Ax \text{ for some } x \in \mathbb{R}^n \} = \text{col}(A).
\]
Let $T : V \to W$ be a linear map.

**Proposition-Definition**

The **image** of $T$ is the set of all function values of $T$, that is,

$$\text{im}(T) = \{w \in W : w = T(v) \text{ for some } v \in V\} \subseteq W.$$ 

Let $A \in M_{mn}(\mathbb{R})$ & $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the assoc linear map. We define

$$\text{im } A = \text{im } T_A = \{b \in \mathbb{R}^m : b = Ax \text{ for some } x \in \mathbb{R}^n\} = \text{col}(A).$$

$\text{im } T$ is a subspace of $W$. 

Remark

Proof omitted, but note we already know $\text{col}(A)$ is a subspace as it is the span of the columns of $A$. 

Daniel Chan (UNSW)
Images of linear maps

Let \( T : V \rightarrow W \) be a linear map.

**Proposition-Definition**

The **image** of \( T \) is the set of all function values of \( T \), that is,

\[
\text{im}(T) = \{ w \in W : w = T(v) \text{ for some } v \in V \} \subseteq W.
\]

Let \( A \in M_{mn}(\mathbb{R}) \) \& \( T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be the assoc linear map. We define

\[
\text{im} A = \text{im} T_A = \{ b \in \mathbb{R}^m : b = Ax \text{ for some } x \in \mathbb{R}^n \} = \text{col}(A).
\]

\( \text{im} T \) is a subspace of \( W \).

**Remark** Proof omitted, but note we already know \( \text{col}(A) \) is a subspace as it is the span of the columns of \( A \).
Verifying whether or not vectors lie in the image

Example

Let 

\[
A = \begin{bmatrix}
1 & 1 \\
0 & 1 \\
1 & 2 \\
\end{bmatrix},
\]

\[
b = \begin{bmatrix}
3 \\
1 \\
3 \\
\end{bmatrix}.
\]

Is \(b \in \text{im} A\)?

Solution

The question amounts to asking: Can we write \(b = Ax\) for some \(x \in \mathbb{R}^2\)?

i.e. Can we solve \(Ax = b\).
Verifying whether or not vectors lie in the image

Example

Let \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}. \)
Verifying whether or not vectors lie in the image

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$. Is $b \in \text{im} \ A$?

Solution

The question amounts to asking: Can we write $b = Ax$ for some $x \in \mathbb{R}^2$? i.e. Can we solve $Ax = b$. 

Daniel Chan (UNSW) 7.4 Subspaces Associated with Linear Maps
Verifying whether or not vectors lie in the image

**Example**

Let \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \), \( b = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \). Is \( b \in \text{im} \ A \)?

**Solution**

*The question amounts to asking: Can we write \( b = Ax \) for some \( x \in \mathbb{R}^2 \)? i.e.*
Verifying whether or not vectors lie in the image

Example

Let \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \), \( b = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \). Is \( b \in \text{im} \ A? \)

Solution

The question amounts to asking: Can we write \( b = Ax \) for some \( x \in \mathbb{R}^2 \)?

i.e. Can we solve \( Ax = b \).
Example

Let \[
\begin{pmatrix}
1 & 2 & 3 & 1 \\
2 & 4 & 7 & 1 \\
1 & 2 & 2 & 2
\end{pmatrix}
\] . Find bases for \( \ker(A) \) and \( \im(A) = \col(A) \).

Solution

\[
\begin{pmatrix}
1 & 2 & 3 & 1 \\
2 & 4 & 7 & 1 \\
1 & 2 & 2 & 2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix} = U
\]

The row echelon form \( U \) has first & third columns leading.
Why study kernels & image?

$\text{im} \ T$ tells you about existence of solutions. $T \ x = b$ has a solution iff $b \in \text{im} \ T$.

$\ker \ T$ tells you about uniqueness of solutions. $T \ x = 0$ has unique solution $x = 0$ iff $\ker \ T = 0$.

We’ll see later, that it also tells you about solutions to $T \ x = b$. 

Daniel Chan (UNSW) 7.4 Subspaces Associated with Linear Maps
Why study kernels & image?

The image $T$ tells you about the existence of solutions. The equation $Tx = b$ has a solution if and only if $b \in \text{im}(T)$.

The kernel $\ker(T)$ tells you about the uniqueness of solutions. The equation $Tx = 0$ has a unique solution $x = 0$ if and only if $\ker(T) = 0$.

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Daniel Chan (UNSW) 7.4 Subspaces Associated with Linear Maps
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\[ \text{im} T \] tells you about existence of solutions.

\[ Tx = b \] has a solution iff \( b \in \text{im} T \).

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\[ \text{im} T \text{ tells you about existence of solutions. } Tx = b \text{ has a solution iff } b \in \text{im} T. \]

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Rank and Nullity

Let $T: V \rightarrow W$ be a linear map & $A$ a matrix with associated linear map $T_A$.

Definition

The nullity of $T$ is $\text{nullity}(T) = \dim \ker(T)$.

The nullity of $A$ is $\text{nullity}(A) = \text{nullity}(T_A) = \dim \ker(A)$.

The rank of $T$ is $\text{rank}(T) = \dim \text{im}(T)$.

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Example (Continued from the example on p.30)

Let \( A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{bmatrix} \).

Find \( \text{nullity}(A) \) and \( \text{rank}(A) \).

Solution

A basis for \( \text{im} A \) was 

Recall a basis for \( \text{ker} A \) had 

Note that the basis vectors for \( \text{im} A \) corresponded to the leading columns of the row-echelon form \( U \) whilst the basis vectors for \( \text{ker} A \) corresponded to the non-leading columns.
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The previous examples illustrate:

**Key Lemma**

Let $A$ be an $m \times n$ matrix which reduces to a row-echelon form $U$.

1. $\text{nullity}(A) = \text{no. parameters in the general soln to } A\mathbf{x} = \mathbf{0} = \text{the number of non-leading columns of } U$.

2. $\text{rank}(A) = \text{the maximal no. independent columns of } A = \text{the number of leading columns of } U$.
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7.4 Subspaces Associated with Linear Maps
The previous examples illustrate

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Rank & Nullity from the row-echelon form
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Our key lemma gives

**Theorem (Rank-nullity Theorem for Matrices)**

If $A$ is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$ 

**Proof.**

This can be used to prove more generally,

**Theorem (Rank-nullity Theorem for Linear Maps)**

Let $T: V \rightarrow W$ be a linear map with $V$ finite dimensional. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$
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Example of rank-nullity theorem

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Let $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the projection map $\text{proj}_b$. 
Example of rank-nullity theorem

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Let \( b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the projection map \( \text{proj}_b \).

Verify the rank-nullity theorem in this case.
Example

Let $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the projection map $\text{proj}_b$.
Verify the rank-nullity theorem in this case.

Solution
Nature of solutions to $Ax = b$

Our key lemma also gives

Theorem

The equation $Ax = b$ has:

1. no solution if $\text{rank} (A) \neq \text{rank} ([A|b])$,
2. at least one solution if $\text{rank} (A) = \text{rank} ([A|b])$.

Further,

i) if $\text{nullity} (A) = 0$, the solution is unique,

ii) if $\text{nullity} (A) = \nu > 0$, then the general solution is of the form

$$x = x_p + \lambda_1 k_1 + \cdots + \lambda_\nu k_\nu$$

for $\lambda_1, \ldots, \lambda_\nu \in \mathbb{R}$, where $x_p$ is any solution of $Ax = b$, and where \{ $k_1, \ldots, k_\nu$ \} is a basis for $\ker (A)$. 
Nature of solutions to $Ax = b$

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1. no solution if $\text{rank}(A) \neq \text{rank}([A | b])$,
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for \( \lambda_1, \ldots, \lambda_\nu \in \mathbb{R} \), where \( \mathbf{x}_p \) is any solution of \( Ax = b \), and where \( \{k_1, \ldots, k_\nu\} \) is a basis for \( \ker(A) \).
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2. at least one solution if \( \text{rank}(A) = \text{rank}([A|b]) \). Further,
   - if \( \text{nullity}(A) = 0 \) the solution is unique, whereas,
   - if \( \text{nullity}(A) = \nu > 0 \), then the general solution is of the form
     \[
     x = x_p + \lambda_1 k_1 + \cdots + \lambda_\nu k_\nu \quad \text{for} \ \lambda_1, \ldots, \lambda_\nu \in \mathbb{R},
     \]
     where \( x_p \) is any solution of \( Ax = b \), and where \( \{ k_1, \ldots, k_\nu \} \) is a basis for \( \ker(A) \).
A theoretical application of rank-nullity theorem

Example

Prove that if \( T : \mathbb{R}^n \to \mathbb{R}^n \) is linear, then the following are equivalent.

a) For all \( \mathbf{b} \in \mathbb{R}^n \), there is at least one solution to \( T\mathbf{x} = \mathbf{b} \).

b) For all \( \mathbf{b} \in \mathbb{R}^n \), there is at most one solution to \( T\mathbf{x} = \mathbf{b} \)
Example

Prove that if $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear, then the following are equivalent.

a) For all $b \in \mathbb{R}^n$, there is at least one solution to $Tx = b$.

b) For all $b \in \mathbb{R}^n$, there is at most one solution to $Tx = b$.

Solution
In what sense are second order linear differential equations linear?

They involve the linear map $T: \mathbb{C}^2[\mathbb{R}] \rightarrow \mathbb{C}[\mathbb{R}]$, where $\mathbb{C}^2[\mathbb{R}]$ is the vector space of all $\mathbb{R}$-valued functions with continuous second derivatives and $\mathbb{C}[\mathbb{R}]$ is the vector space of all continuous $\mathbb{R}$-valued functions—

$$T(y) = ay'' + by' + cy,$$

where $a, b, c \in \mathbb{R}$.

In this case, $\ker(T)$ is the solution set of the ODE 

$$a y'' + by' + cy = 0,$$

where $a, b, c \in \mathbb{R}$.

Hence the homogeneous solution is a vector space. Furthermore, it is of dimension 2 i.e. $\text{nullity}(T) = 2$.

We can also apply similar ideas for the solution to $A x = b$ to get the solution to 

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They involve the linear map $T: C^2\mathbb{R}\rightarrow C\mathbb{R}$, where $C^2\mathbb{R}$ is the vector space of all $\mathbb{R}$-valued functions with continuous second derivatives and $C\mathbb{R}$ is the vector space of all continuous $\mathbb{R}$-valued functions.

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We can also apply similar ideas for the solution to $Ax = b$ to get the solution to $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$, where $a, b, c \in \mathbb{R}$.
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Linear Differential Equations

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Hence the homogeneous solution is a vector space. Furthermore, it is of dimension 2 i.e. $\text{nullity}(T) = 2$. We can also apply similar ideas for the solution to $Ax = b$ to get the solution to

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Example involving polynomials

To study boundary value problems, it's useful to study linear maps such as the one below.

Example

The function $T : \mathbb{P}_2 \rightarrow \mathbb{R}^2$ is defined by

$$T(p) = (p(1), p'(1))$$

a) Prove that $T$ is linear.

b) Find $\ker(T)$.

c) Use the rank-nullity theorem to find $\text{im}(T)$.

Solution

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Solution (Continued)