Chapter 5: Matrices

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In this chapter

Matrices were first introduced in the Chinese “Nine Chapters on the Mathematical Art” to solve linear eqns. In the mid-1800s, senior wrangler (see wikipedia) Arthur Cayley studied matrices in their own right and showed how they have an interesting and useful algebra associated to them. We will look at Cayley’s ideas and extend vector arithmetic to matrices and even show there is matrix multiplication akin to multiplying numbers. These ideas will not only shed light on solving linear eqns, they will also be useful later when you look at multivariable functions and mappings.
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These ideas will not only shed light on solving linear eqns, they will also be useful later when you look at multivariable functions and mappings.
Some new notation for matrices

Recall an $m \times n$-matrix is an array of (for us) scalars (real or complex).

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}.
\]

Notation

We abbreviate the above to $A = (a_{ij})$ and call $a_{ij}$ the $ij$-th entry of $A$.

Also write $[A]_{ij}$ for $a_{ij}$.

We say the size of $A$ is $m \times n$ because it has $mn$ ($\mathbb{R}$) (resp $mn$ ($\mathbb{C}$)) denote the set of all $m \times n$-matrices with real entries (resp complex entries). Sometimes abbreviate to $M_{mn}$ if the scalars are understood or irrelevant.

E.g. A length $m$ column vector is an...
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Let $A = (a_{ij}) = (a_1 | a_2 | ... | a_n) \in M_{mn}$. Then

\[
A \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = x_1 a_{11} + x_2 a_{22} + ... + x_n a_{nn}.
\]

Alternatively, the $i$-th entry of $Ax$ is $[Ax]_i = a_{i1} x_1 + ... + a_{in} x_n = (a_{i1} ... a_{in}) \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}$.

Note similarity with dot products. $A$ induces the linear function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m: x \mapsto Ax$. Note we will write all our results for matrices with real entries, but there are obvious analogues over the complexes.
Let $A = (a_{ij}) = (a_1|a_2| \ldots |a_n) \in M_{mn}$. Then
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**Note** We will write all our results for matrices with real entries, but there are obvious analogues over the complexes.
Arithmetic of matrices

Just as for vectors, we can define matrix addition and scalar multiplication to be entry-wise addition and scalar multiplication.

E.g.\[
\begin{pmatrix}
1 & 2 & 3 \\
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\end{pmatrix} +
\begin{pmatrix}
3 & 4 & 6 \\
2 & -1 & 5 \\
\end{pmatrix} =
\begin{pmatrix}
7 & 6 & 5 \\
2 & 4 & 4 \\
\end{pmatrix}
\]

In formulas:

Matrix arithmetic

For \(A, B \in M_{mn}(\mathbb{R})\), \(\lambda \in \mathbb{R}\), the entries of \(A + B\), \(\lambda A\) \(\in M_{mn}(\mathbb{R})\) are

\[
[A + B]_{ij} = [A]_{ij} + [B]_{ij}
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[\lambda A]_{ij} = \lambda [A]_{ij}
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N.B. We don't define the sum of matrices of different sizes (just as is the case for vectors).
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Linear combinations and subtraction

E.g. We can also form linear combinations of matrices 

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Definition

The zero matrix \(0\) has all entries 0. (There's one for each size \(m \times n\).)

\[A + 0 = \]

The negative of \(A\) \(\in \mathbb{M}_{mn}\) is \(-A := (-1)A\).

Hence \(A + (-A) = \)

The difference \(A - B = A + (-B)\) if \(A, B\) have the same size.
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- The difference $A - B = A + (-B)$ if $A, B$ have the same size.
Another distributive & associative law

Proposition

For $A, B \in M_{mn}(\mathbb{R}), \lambda \in \mathbb{R}, x \in \mathbb{R}^n$

Proof. Suppose $n = 2$ (else need more space) so $A = (a_1 | a_2)$, $B = \ldots$

Upshot

Recall that in calculus, you define the sum and scalar multiple of functions pointwise, $(f + g)(x) = f(x) + g(x)$, $(\lambda f)(x) = \lambda f(x)$.

The above formulas show that the linear function corresponding to $A + B$ which sends $x \mapsto (A + B)x = Ax + Bx$ is the pointwise sum of the functions corresponding to $A$ and $B$. The same goes for the scalar multiple.
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The above formulas show that the linear function corresponding to $A + B$ which sends $x \mapsto (A + B)x = Ax + Bx$ is the pointwise sum of the functions corresponding to $A$ and $B$. The same goes for the scalar multiple.
E.g. The mapping $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ corresponding to the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

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Basic properties of matrix arithmetic

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For \( A, B \in \mathbb{M}_{mn} \), and scalars \( \lambda, \mu \),

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\lambda (\mu A) = (\lambda \mu) A
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\[
(\lambda + \mu) A = \lambda (A + B)
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Proof.

Just as for vectors e.g.
<table>
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Basic properties of matrix arithmetic

**Proposition**

For $A, B \in M_{mn}$, and scalars $\lambda, \mu$

- $\lambda(\mu A) = (\lambda \mu)A$.
- $(\lambda + \mu)A =$
- $\lambda(A + B) =$

**Proof.** Just as for vectors e.g.
Matrix multiplication

Let \( A \in M_{mn} \), \( B = (b_1 | \ldots | b_p) \in M_{np} \).

We define the matrix product \( AB \) to be the \( m \times p \)-matrix
\[
AB = (A b_1 | \ldots | A b_p) \in M_{mp}.
\]

E.g.
\[
\begin{pmatrix}
0 & 1 \\
2 & 3 \\
4 & 5
\end{pmatrix}
(1 - 1 1 2)
\]

Alternatively, the \( ij \)-th entry of \( AB \) comes from "zipping up" the \( i \)-th row of \( A \) with the \( j \)-th column of \( B \): i.e. if \( A = (a_{ij}) \), \( B = (b_{ij}) \)
\[
[AB]_{ij} = (a_{i1} \ldots a_{in}) \begin{pmatrix}
 b_{1j} \\
\vdots \\
b_{nj}
\end{pmatrix} = a_{i1} b_{1j} + \ldots + a_{in} b_{nj} = \sum_{l=1}^{n} a_{il} b_{lj}.
\]

Warning
The product \( AB \) is only defined when no. columns \( A \) = no. rows \( B \).
Matrix multiplication

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Warning The product $AB$ is only defined when no. columns $A = $ no. rows $B$. 
Associative law

Let $A \in M_{mn}$, $B = (b_1 | \ldots | b_p) \in M_{np}$, $C = (c_1 | \ldots | c_q) \in M_{pq}$.

Then $(AB)C = A(BC)$.

Proof. It suffices to show this for $C = \begin{pmatrix} c_1 & \ldots & c_p \end{pmatrix}$.

Assuming this case we see $(AB)c = (A(b_1) | \ldots | A(b_p))c = A((b_1)c_1 + \ldots + (b_p)c_p) = A((Bc_1) | \ldots | (Bc_p))$. If $C = c$ then $(AB)c = (A(b_1) | \ldots | A(b_p))c = c_1A(b_1) + \ldots + c_pA(b_p) = A((c_1b_1) + \ldots + (c_pb_p)) = A(BC)$. 

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Associative law

Associative law of matrix multiplication

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Chapter 5: Matrices
Semester 1 2017
Associative law of matrix multiplication

Let \( A \in M_{mn}, B = (b_1 | \ldots | b_p) \in M_{np}, C = (c_1 | \ldots | c_q) \in M_{pq} \). Then \((AB)C = A(BC)\).
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**Associative law of matrix multiplication**

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\[
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= A((Bc_1)|\ldots|(Bc_q)) = A(BC).
\]

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\[
(AB)c = (Ab_1|\ldots|Ab_p)c = c_1Ab_1 + \ldots + c_pAb_p = A(c_1b_1 + \ldots + c_pb_p) = A(BC).
\]
Functional interpretation of the associative law

The associative law says the function associated to \( AB \) which maps \( x \mapsto (AB)x \) is the composite \( x \mapsto Bx \mapsto A(Bx) \) of the linear maps associated to \( A \) and \( B \).

E.g. Recall that \( B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) corresponds to reflection about the \( x \)-axis.

Let’s check \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Remark: The definition of matrix multiplication was designed so that it reflects the composition of linear maps.
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E.g. Recall that $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ corresponds to reflection about the $x$-axis. The functional viewpoint shows that $B^2$ corresponds to the mapping $(1 0) (1 0) =-1$.
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**Remark** The definition of matrix multiplication was designed so that it reflects the composition of linear maps.
Distributive laws & noncommutativity

Distributive law

Let $A$, $B$, $C$ be matrices & $\lambda$ a scalar. The following formulas hold whenever the terms on one side are defined.

1. $A(B + C) = AB + AC$.
2. $(A + B)C = AC + BC$.
3. $(\lambda A)B = \lambda (AB) = A(\lambda B)$.

Proof. Easy ex similar to distributive law we proved for matrix-vector product.

Noncommutativity

Note that if $AB$ is defined, $BA$ may not be, and even if it is, usually we have $AB \neq BA$.

E.g. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Hence $(A + B)^2 = \ldots$
Distributive laws & noncommutativity

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Hence $(A + B)^2 = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = A^2$. 

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Distributive laws & noncommutativity

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E.g. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Hence $(A + B)^2 =$
A matrix is said to be *square* if its no. rows = no. columns.
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**E.g.**

**Formula**

$$IA = A, BI = B$$ whenever the products are defined.
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**Proof.** Just multiply matrices. We'll check here
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### Formula

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**Proof.** Just multiply matrices. We'll check here

**Upshot** In particular, we see the linear function associated to \( I \) is the identity map \( x \mapsto x \).
Transpose

The transpose of an \( m \times n \)-matrix \( A \), is the \( n \times m \)-matrix \( A^T \) gotten by turning all the rows of \( A \) into columns (or equivalently, flipping the matrix about the row-column diagonal).

E.g. \[
\begin{pmatrix}
2 & 5 & 1 \\
1 & 3 & 2
\end{pmatrix}^T
\]

More formally, the entries of \( A^T \) are given by \( A^T_{ij} = A_{ji} \).

Formulas

Let \( A, B \) be matrices & \( \lambda \) a scalar. The following hold when one side is defined.

1. \( (A^T)^T = A \)
2. \( (A + B)^T = A^T + B^T \)
3. \( (\lambda A)^T = \lambda A^T \)

Proof.

1) & 2) are easy and say the function \( A \mapsto A^T \) is linear. For 3) \[
\left[(B^T A^T)\right]_{ij} = \sum_l [B^T]_{il} [A^T]_{lj} = \sum_l [B]_{li} [A]_{jl} = \left[(AB)^T\right]_{ij}
\]
The *transpose* of an $m \times n$-matrix $A$, is the $n \times m$-matrix $A^T$ gotten by turning all the rows of $A$ into columns (or equivalently, flipping the matrix about the row $i = \text{column } j$ diagonal).
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^T
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\begin{pmatrix}
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\end{pmatrix}^T = \]

More formally, the entries of \( A^T \) are given by \( (A^T)_{ij} = [A]_{ji} \).

**Formulas**

Let \( A, B \) be matrices & \( \lambda \) a scalar. The following hold when one side is defined.

1. \((A^T)^T = A\)
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**Formulas**

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3. \((AB)^T = B^T A^T\).
The transpose of an $m \times n$-matrix $A$, is the $n \times m$-matrix $A^T$ gotten by turning all the rows of $A$ into columns (or equivalently, flipping the matrix about the row $i = \text{column } j$ diagonal).

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More formally, the entries of $A^T$ are given by $[A^T]_{ij} = [A]_{ji}$.

### Formulas

Let $A, B$ be matrices & $\lambda$ a scalar. The following hold when one side is defined.

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$, \hspace{1em} $(\lambda A)^T = \lambda A^T$
3. $(AB)^T = B^T A^T$.

**Proof.** 1) & 2) are easy and say the function $A \mapsto A^T$ is linear. For 3)

$$[(B^T A^T)]_{ij} = \sum_l [B^T]_{il} [A^T]_{lj} = \sum_l [B]_{li} [A]_{lj} =$$
Miscellaneous tidbits involving transpose

Relation with dot product

Let $a, b \in \mathbb{R}^n$. Then though $ab$ is not defined we can define $a^T b = (a_1 \ldots a_n) \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \ldots + a_n b_n = a \cdot b$.

Symmetric matrices

A square matrix is symmetric if $A^T = A$ and anti-symmetric if $A^T = -A$ (Why is square in the defn?).

E.g. $b b^T \in M_{nn}$ is symmetric since $b b^T x = \|b\|^2 x$ for all $x \in \mathbb{R}^n$.
Miscellaneous tidbits involving transpose

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Symmetric matrices

A square matrix is symmetric if $A^T = A$ and anti-symmetric if $A^T = -A$.

E.g. $b b^T \in M_{nn}$ is symmetric since $|b| = 1$.

In fact if $|b| = 1$, then the linear function $\mathbb{R}^n \rightarrow \mathbb{R}^n$ associated to $b b^T$ is projection onto $b$ for it sends $x \mapsto b b^T x$. 
Miscellaneous tidbits involving transpose

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Let \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \).

\[
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a_1 \\ \vdots \\ a_n
\end{pmatrix}
\begin{pmatrix}
b_1 \\ \vdots \\ b_n
\end{pmatrix}
= a_1 b_1 + \cdots + a_n b_n
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Miscellaneous tidbits involving transpose

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Relation with dot product
Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Then though $\mathbf{a}\mathbf{b}$ is not defined we can define

$$\mathbf{a}^T \mathbf{b} = (a_1 \ldots a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \ldots + a_n b_n = \mathbf{a} \cdot \mathbf{b}.$$
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A square matrix is *symmetric* if \( \mathbf{A}^T = \mathbf{A} \) and *anti-symmetric* if \( \mathbf{A}^T = -\mathbf{A} \).
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Let \( a, b \in \mathbb{R}^n \). Then though \( ab \) is not defined we can define

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Miscellaneous tidbits involving transpose

Relation with dot product
Let \( a, b \in \mathbb{R}^n \). Then though \( ab \) is not defined we can define

\[
a^T b = (a_1 \ldots a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \ldots + a_n b_n = a \cdot b.
\]

Symmetric matrices
A square matrix is *symmetric* if \( A^T = A \) and *anti-symmetric* if \( A^T = -A \) (Why is square in the defn?).

E.g. \( bb^T \in M_{nn} \) is symmetric since
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E.g. \( \mathbf{b} \mathbf{b}^T \in M_{nn} \) is symmetric since

In fact if \( |\mathbf{b}| = 1 \), then the linear function \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) associated to \( \mathbf{b} \mathbf{b}^T \) is projection onto \( \mathbf{b} \).
Relation with dot product

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x \mapsto bb^T x =
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Matrix Inverse

To attempt matrix "division" we need the matrix inverse.

A matrix \( W \) is an inverse for the matrix \( A \) if \( AW = WA = I \).

If such a matrix exists we say that \( A \) is invertible.

A matrix can have at most one inverse, denoted \( A^{-1} \).

Proof.

E.g. \( A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \), \( B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \).
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Let $A \in \mathbb{M}_{mn}$ be an invertible matrix. Then for any $b \in \mathbb{R}^n$ the eqn $Ax = b$ has $x = A^{-1}b$ as its unique soln.

Proof. $x = A^{-1}b$ is a soln since $Ax = b = \Rightarrow$

Corollary

Any invertible matrix $A \in \mathbb{M}_{mn}$ is square.

Proof. The inverse $W = A^{-1}$ must be $n \times m$ for $AW, WA$ to be defined. $A^{-1}x = 0$ has a unique soln $\Rightarrow$
Inverses and solving $Ax = b$

**Proposition**

Let $A \in M_{mn}$ be an invertible matrix.

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$$Ax = b \Rightarrow \text{It is the only possible solution since } A^{-1}b$$

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Formulas involving matrix inverses

Let $A, B \in \mathbb{M}_{n \times n}$ be invertible.

1. $A^{-1}$ is invertible & \((A^{-1})^{-1} = A\).

2. $AB$ is invertible & \((AB)^{-1} = B^{-1}A^{-1}\).

3. $A^T$ is invertible & \((A^T)^{-1} = (A^{-1})^T\).

Proof. All easy e.g. Simplify \((ABA^{-1})^{-2} A\).
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Computing inverses of matrices: example

Find the inverse of 

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 3 & 2 \\
-1 & 1 & -1
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The algorithm for inverting matrices

Let $A \in M_{n \times n}$. To determine invertibility and invert $A$ if invertible we:

1. Form the augmented $n \times 2n$-matrix $\begin{pmatrix} A & I \end{pmatrix}$.
2. Apply ERO’s until we get row echelon form $\begin{pmatrix} U & B \end{pmatrix}$.
3. If $U$ has non-leading columns, we stop as solns to $Ax = 0$ are not unique so $A$ is not invertible.
4. If $U$ has no non-leading columns, then $A$ is invertible & we can apply EROS to transform $\begin{pmatrix} U & B \end{pmatrix}$ to the form $\begin{pmatrix} I & C \end{pmatrix}$ with $C = A^{-1}$.

Is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 4 & 3 \end{pmatrix}$ invertible? If so, find its inverse.
Let $A \in M_{nn}$. To determine invertibility and invert $A$ if invertible we:

1. Form the augmented $n \times 2n$-matrix $(A | I)$.
2. Apply ERO’s until we get row echelon form $(U | B)$.
3. If $U$ has non-leading columns, we stop as solns to $A x = 0$ are not unique so $A$ is not invertible.
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Why our algorithm works

Suppose $EROs \rightarrow (I|C)$ where $C = (c_1|...|c_n)$.

Restrict Gaussian elimination $\rightarrow (I|c_i)$, i.e. $Ac_i = e_i$.

$AC = (Ac_1|...|Ac_n) = (e_1|...|e_n) = ??

Challenge Q
Show that we also have $CA = I$.

Hint: Observe $ACA = A = AI$ and explain why you can cancel the $A$'s.
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Invertibility for square matrices

Theorem
Suppose that $A \in \mathbb{M}_{nn}$ has row-echelon form $U$. Then the following are equivalent:

1. $A$ is invertible.
2. $U$ has no zero rows.
3. $U$ has no nonleading columns.
4. $A \mathbf{x} = \mathbf{0}$ has a unique solution $\mathbf{x} = \mathbf{0}$.
5. For each $b \in \mathbb{R}^n$, $A \mathbf{x} = b$ has a unique solution.

Proof
We check $5) \Rightarrow 4) \Rightarrow 3) \Rightarrow 2) \Rightarrow 1) \Rightarrow 5)$.

Corollary
Suppose that $A \in \mathbb{M}_{nn}$. Then $X A = I \iff A X = I \iff X = A^{-1}$.

Proof.
Note $A X$ or $X A$ square $\Rightarrow X \in \mathbb{M}_{nn}$, so by symmetry, suff show $A X = I \Rightarrow X$ is invertible by checking condn 4) above. But $X \mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = I \mathbf{x} = A X \mathbf{x} = A \mathbf{0} = \mathbf{0}$ so we're done by the thm.
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Suppose that $A \in M_{nn}$. Then

$XA = I \iff AX = I \iff X = A^{-1}$.

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Note $AX$ or $XA$ square $\Rightarrow X \in M_{nn}$, so by symmetry, suffice to show $AX = I \Rightarrow X$ is invertible by checking condn 4) above.

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Theorem

Suppose that $A \in M_{nn}$ has row-echelon form $U$. Then the following are equivalent:

1. $A$ is invertible.
2. $U$ has no zero rows.
3. $U$ has no nonleading columns.
4. $Ax = 0$ has a unique solution $x = 0$.
5. For each $b \in \mathbb{R}^n$, $Ax = b$ has a unique solution.

Proof. We check 5) $\implies$ 4) $\implies$ 3) $\implies$ 2) $\implies$ 1) $\implies$ 5)

Corollary Suppose that $A \in M_{nn}$. Then

$$XA = I \iff AX = I \iff X = A^{-1}.$$ 

Proof. Note $AX$ or $XA$ square $\implies X \in M_{nn}$, so by symmetry, suff show $AX = I \implies X$ is invertible by checking condn 4) above. But $Xx = 0 \implies x = Ix = AXx = A0 = 0$ so we’re done by the thm.
Inverse of $2 \times 2$-matrices

Recall the $2 \times 2$-determinant:

$$
\begin{vmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{vmatrix} = \det(A) = a_{11} a_{22} - a_{12} a_{21}.
$$

Inverse formula

Let $A = (a_{ij}) \in M_{2,2}$. Then $A$ is invertible if $\det(A) \neq 0$ in which case $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix}
  a_{22} & -a_{12} \\
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\end{pmatrix}$.

Proof. Just multiply

Remark

We'll generalise the relationship between determinants and invertibility later. The formula in this case is best appreciated if you understand the linear mapping $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ associated to $A$. 

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Minors of a square matrix

We define the determinant of $A \in M_{nn}$, using induction on $n$.

We've defined it for $n = 2$ (and 3), and suppose it's defined for $(n-1) \times (n-1)$-matrices. First

Definition

For $1 \leq i, j \leq n$, the $(row \ i, \ column \ j)$ minor of $A$, denoted $|A|_{ij}$, is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row $i$ and column $j$ from $A$.

E.g. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 4 & -2 & 5 \end{pmatrix}$.

$|A|_{21} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$.

Defn: For a $1 \times 1$-matrix $(a)$ we define its determinant to be $\text{det}(a) = a$.

There are many definitions of the determinant. To keep the exposition elementary, we'll use a generalisation of the one for $3 \times 3$-determinants.
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Definition
Let $A \in \mathbb{M}_{n \times n}$ with $n \geq 2$. Then the determinant of $A$ is

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\det(A) = a_{11} |A_{11}| - a_{12} |A_{12}| + a_{13} |A_{13}| + \cdots + (-1)^{n-1} a_{1n} |A_{1n}|.
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E.g. Let $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ -1 & 1 & 5 & 0 \\ 7 & -2 & 3 & 1 \end{pmatrix}$.

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Determinant of lower triangular matrices

We say $A = (a_{ij})$ is lower triangular, if $a_{ij} = 0$ whenever $i < j$.

The computation above shows more generally the following Proposition

If $A = (a_{ij})$ is lower triangular, then $\det(A)$ is just the product of the diagonal elements.

$\det(A) = \prod_{i=1}^{n} a_{ii}$

In particular, $\det(I) = 1$. 
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Unfortunately in this course, we will not be able to talk about the determinant in depth. For a thorough treatment, see my MATH2601 lecture notes, lectures 5, 6.

Just as the $2 \times 2$-determinant gives the area of parallelograms, and $3 \times 3$-determinants volumes of parallelopipeds, the $n \times n$-determinant gives the volume of $n$-dimensional parallelopipeds (unfortunately we can't prove this).

In particular, the linear mapping $\mathbb{R}^n \to \mathbb{R}^n$ associated to $A$ expands volumes by $\det(A)$.

The sign of $\det(A)$ is related to a higher dimensional version of the right hand rule.

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Determinant of the Transpose

Let $A = (a_{ij}) \in M_{nn}$. Then

**Proposition** $\det(A) = \det(A^T)$.

**Proof.** Hard (omitted) with our definition except for $2 \times 2$.

Since the transpose swaps columns with rows, we can compute $\det(A)$ by expanding along the first column as in the formula below:

$$\det(A) = a_{11} |A_{11}| - a_{21} |A_{21}| + \cdots + (-1)^{n+1} a_{n1} |A_{n1}|.$$

In particular, if $A$ is upper triangular in the sense that $a_{ij} = 0$ whenever $i > j$, then $\det(A)$ is the product of the diagonal entries. E.g.
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**E.g.**
Determinant of products

Theorem

\[
\text{det}(AB) = \text{det}(A) \text{det}(B)
\]

Note that \(A, B\) square and \(AB\) defined \(\Rightarrow\) \(A, B\) have the same size.

Proof.

Hard, see my MATH2601 notes.

E.g. Find the determinant of \(A\) where \(A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & e^{\pi} - \sqrt{2} & 1 \\ 0 & 0 & \int_1^0 dx \sqrt{(1 - x^2)(2 - x^2)} \end{pmatrix}\).

Corollary

If \(A\) is invertible, then \(\text{det}(A) \neq 0\) & \(\text{det}(A^{-1}) = \frac{1}{\text{det}(A)}\).

Proof.

Daniel Chan (UNSW)

Chapter 5: Matrices

Semester 1 2017
Theorem

\[ \det(AB) = \det(A) \det(B) \]

Note that \( A, B \) square and \( AB \) defined \( \Rightarrow \) \( A, B \) have the same size.

Proof. Hard, see my MATH2601 notes.

E.g. Find the determinant of \( A \)

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\begin{bmatrix}
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Proof. Daniel Chan (UNSW)
Determinant of products

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**E.g.** Find the determinant of \( A^{42} \) where \( A = \begin{pmatrix} 1 & 0 & 0 \\ 4e^{\pi - \sqrt{2}} & \frac{1}{2} & 0 \\ 0 & \int_0^1 \frac{dx}{\sqrt{(1-x^2)(2-x^2)}} & 4 \end{pmatrix} \).
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Proof.
Below we see how performing an ERO to get from $A \rightarrow B$ changes the determinant.

**Proposition 1**

If $A \leftrightarrow B$, then $\det(B) = -\det(A)$. i.e. swapping two rows of a matrix negates the determinant.

**Proposition 2**

If $A \rightarrow_{R_i} cR_i$, then $\det(B) = c\det(A)$. i.e. multiplying a row by $c$ multiplies the determinant by $c$.

**Proposition 3**

If $A \rightarrow_{R_i} R_i + cR_j$, then $\det(B) = \det(A)$. i.e. In particular, if two rows of $A$ are the same, or $A$ has a row of 0s, then $\det(A) = 0$.

E.g. Suppose $A \in \mathbb{M}_{44}$ has determinant 2. Find $\det(3A)$. 
Below we see how performing an ERO to get from $A \xrightarrow{ERO} B$ changes the determinant.

**Proposition 1**
If $\leftrightarrow_{R_{ij}} A \xrightarrow{ERO} B$, then $\det(B) = -\det(A)$. i.e. swapping two rows of a matrix negates the determinant.

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If $\leftrightarrow_{R_i = cR_i} A \xrightarrow{ERO} B$, then $\det(B) = c\det(A)$, i.e., multiplying a row by $c$ multiplies the determinant by $c$.

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If $\leftrightarrow_{R_i = R_i + cR_j} A \xrightarrow{ERO} B$, then $\det(B) = \det(A)$. i.e., in particular, if two rows of $A$ are the same, or $A$ has a row of 0s, then $\det(A) = 0$.

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Daniel Chan (UNSW)
Chapter 5: Matrices
Semester 1 2017
EROs and determinants

Below we see how performing an ERO to get from \( A \xrightarrow{ERO} B \) changes the determinant.

### Proposition

1. If \( A \xrightarrow{R_i \leftrightarrow R_j} B \) then \( \det(B) = -\det(A) \).

### Example

Suppose \( A \in \mathbb{M}_{44} \) has determinant 2. Find \( \det(3A) \).
EROs and determinants

Below we see how performing an ERO to get from $A \xrightarrow{ERO} B$ changes the determinant.

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EROs and determinants

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**E.g.** Suppose $A \in M_{44}$ has determinant 2. Find $\det(3A)$. 

Computing determinants in practice

Warning
We don’t compute determinants using the definition except for small or special matrices!

In practice,

a) we apply EROs to reduce $A$ to row-echelon form $U$,
b) record how we’ve changed the determinant (see previous slide),
& c) note $U$ square & row-echelon $\Rightarrow$ upper triangular so has readily computable determinant.

E.g. Find the determinant of

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 1 & 1 & 2 \\ -1 & 2 & 0 & 1 \\ 2 & 0 & 3 & -1 \end{bmatrix}$$
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Conclusions from the algorithm for computing determinants

We can transform $A$ into upper echelon form $U$ by using ERO's of form $R_i \leftrightarrow R_j$ and $R_i = R_i + cR_j$ only.

Then $\det(A) = \pm \det(U)$.

For this $U$, it has all leading columns if and only if all diagonal entries are non-zero if and only if $\det(U) \neq 0$.

This gives

Theorem

A square matrix $A$ is invertible if and only if $\det(A) \neq 0$.

Remark

The formula for inverting $2 \times 2$-matrices generalises (see Cramer's rule in my MATH2601 notes). It has $\det(A)$ in the denominator for $A^{-1}$ and an otherwise well-defined numerator.
Conclusions from the algorithm for computing determinants

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Then $\det(A) = \pm \det(U)$. For this $U$, it has all leading columns iff all diagonal entries are non-zero iff $\det(U) \neq 0$. This gives

Theorem

A square matrix $A$ is invertible iff $\det(A) \neq 0$.

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Conclusions from the algorithm for computing determinants

- We can transform $A$ into upper echelon form $U$ by using ERO’s of form $R_i \leftrightarrow R_j \ & R_i = R_i + cR_j$ only. Then det($A$) = ± det($U$).
- For this $U$, it has all leading columns iff

\[ \text{det}(U) \neq 0. \]
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