In this chapter

Matrices were first introduced in the Chinese "Nine Chapters on the Mathematical Art" to solve linear eqns. In the mid-1800s, senior wrangler (see wikipedia) Arthur Cayley studied matrices in their own right and showed how they have an interesting and useful algebra associated to them. We will look at Cayley's ideas and extend vector arithmetic to matrices and even show there is matrix multiplication akin to multiplying numbers. These ideas will not only shed light on solving linear eqns, they will also be useful later when you look at multivariable functions and mappings.
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Some new notation for matrices

Recall an $m \times n$-matrix is an array of (for us) scalars (real or complex).

$$A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
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Notation

- We abbreviate the above to $A = (a_{ij})$ and call $a_{ij}$ the $ij$-th entry of $A$. 
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**E.g.** A length $m$ column vector is an
Revise matrix-vector product

Let $A = (a_{ij}) = (a_1 | a_2 | ... | a_n) \in M_{mn}$. Then

$$A \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + ... + x_n a_n.$$

Alternatively, the $i$-th entry of $A x$ is $[A x]^i = a_{i1} x_1 + ... + a_{in} x_n = (a_{i1} ... a_{in}) \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}$.

Note similarity with dot products.

$A$ induces the linear function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$: $x \mapsto A x$.

Note: We will write all our results for matrices with real entries, but there are obvious analogues over the complexes.
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Arithmetic of matrices

Just as for vectors, we can define matrix and scalar multiplication to be entry-wise addition and scalar multiplication.

E.g.
\[
\begin{pmatrix}
1 & 2 & 3 \\
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\end{pmatrix} +
\begin{pmatrix}
3 & 4 & 6 \\
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\begin{pmatrix}
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\]

In formulas:

**Matrix arithmetic**

For \(A, B \in M_{mn}(\mathbb{R})\), \(\lambda \in \mathbb{R}\), the entries of \(A + B\), \(\lambda A\) ∈ \(M_{mn}(\mathbb{R})\) are:

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[A + B]_{ij} = [A]_{ij} + [B]_{ij} \quad \lambda [A]_{ij} = \lambda [A]_{ij}
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N.B. We don't define the sum of matrices of different sizes (just as is the case for vectors).
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= 
\begin{pmatrix}
7 & 6 & 10 \\
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Linear combinations and subtraction

E.g. We can also form linear combinations of matrices

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Definition

The zero matrix \( 0 \) has all entries 0. (There's one for each size \( m \times n \).)

\[ A + 0 = A \]

The negative of \( A \in M_{mn} \) is \( -A := (\begin{array}{c}
-1 \\
& 1 & 2 & 3 \\
\end{array}) \).

Hence \( A + (-A) = 0 \).

The difference \( A - B = A + (-B) \) if \( A, B \) have the same size.
**Example.** We can also form linear combinations of matrices

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- The *difference* \(A - B = A + (-B)\) if \(A, B\) have the same size.
Another distributive & associative law

Proposition

For $A, B \in M_{mn}(\mathbb{R}), \lambda \in \mathbb{R}, x \in \mathbb{R}^n$,

$$(A + B)x = Ax + Bx,$$

$$(\lambda A)x = \lambda (Ax).$$

Proof. Suppose $n = 2$ (else need more space) so $A = (a_1 | a_2), B = \ldots$. Recall that in calculus, you define the sum and scalar multiple of functions pointwise, $(f + g)(x) = f(x) + g(x), (\lambda f)(x) = \lambda f(x)$.

The above formulas show that the linear function corresponding to $A + B$ which sends $x \mapsto (A + B)x = Ax + Bx$ is the pointwise sum of the functions corresponding to $A$ and $B$. The same goes for the scalar multiple.
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Mappings and matrix arithmetic

E.g. The mapping $\mathbb{R}^2 \to \mathbb{R}^2$ corresponding to the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

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Basic properties of matrix arithmetic

Proposition

For $A, B \in M_{mn}$, and scalars $\lambda, \mu$

$$\lambda (\mu A) = (\lambda \mu) A$$

$$(\lambda + \mu) A = \lambda (A + B)$$

Proof.

Just as for vectors e.g.
Basic properties of matrix arithmetic

**Proposition**

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Basic properties of matrix arithmetic

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**Proof.** Just as for vectors e.g.
Matrix multiplication

Let $A \in M_{mn}$, $B = \begin{pmatrix} b_1 & \ldots & b_p \end{pmatrix} \in M_{np}$. We define the matrix product $AB$ to be the $m \times p$-matrix $AB = \begin{pmatrix} A \cdot b_1 & \ldots & A \cdot b_p \end{pmatrix}$.

E.g. $\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{pmatrix} \left( \begin{array}{c} 1 \\ -1 \\ 1 \\ 2 \end{array} \right)$

Alternatively, the $ij$-th entry of $AB$ comes from “zipping up” the $i$-th row of $A$ with the $j$-th column of $B$:

\[ (AB)_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}. \]

Warning: The product $AB$ is only defined when no. columns $A =$ no. rows $B$. 

Daniel Chan (UNSW) 
Chapter 5: Matrices 
Semester 1 2015
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$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{pmatrix} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) = \begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 4 & -4 \end{pmatrix}$$

Alternatively, the $ij$-th entry of $AB$ comes from "zipping up" the $i$-th row of $A$ with the $j$-th column of $B$: i.e. if $A = (a_{ij})$, $B = (b_{ij})$, 

$$[AB]_{ij} = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix} = a_{i1} b_{1j} + \ldots + a_{in} b_{nj} = \sum_{l=1}^{n} a_{il} b_{lj}.$$ 

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Chapter 5: Matrices 
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**Warning** The product $AB$ is only defined when no. columns $A = \text{no. rows } B$. 
Associative law

Associative law of matrix multiplication

Let \( A \in \mathbb{M}_{mn} \), \( B = (b_1 | ... | b_p) \in \mathbb{M}_{np} \), \( C = (c_1 | ... | c_q) \in \mathbb{M}_{pq} \).

Then \((AB)C = A(BC)\).

Proof.
It suffices to show this for \( C = c = \begin{pmatrix} c_1 \\ ... \\ c_p \end{pmatrix} \) for assuming this case we see

\((AB)c = \begin{pmatrix} (AB)c_1 \\ ... \\ (AB)c_p \end{pmatrix} = 
\begin{pmatrix} A(Bc_1) \\ ... \\ A(Bc_p) \end{pmatrix} = A(BC)\).

If \( C = c \) then

\((AB)c = (A b_1 | ... | A b_p) c = c_1 A b_1 + ... + c_p A b_p = A (c_1 b_1 + ... + c_p b_p) = A(BC)\).
Associative law of matrix multiplication

Let $A \in M_{mn}$, $B = (b_1 \ldots b_p) \in M_{np}$, $C = (c_1 \ldots c_q) \in M_{pq}$.

Then $(AB)C = A(BC)$.

Proof. It suffices to show this for $C = \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix}$ for assuming this case we see

$$(AB)c_1 = (A(b_1|\ldots|b_p))c_1 = (A(c_1))c_1 + \ldots + (A(c_p))c_p = A(c_1b_1 + \ldots + c_pb_p) = A(BC).$$
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Daniel Chan (UNSW)
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Functional interpretation of the associative law

The associative law says the function associated to $AB$ which maps $x \mapsto (AB)(x) = A(Bx)$ is the composite $x \mapsto Bx \mapsto A(Bx)$ of the linear maps associated to $A$ and $B$.

E.g. Recall that $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ corresponds to reflection about the $x$-axis.

Let's check $(1 0)(1 0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Remark: The definition of matrix multiplication was designed so that it reflects the composition of linear maps.
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Distributive laws & noncommutativity

Let $A$, $B$, $C$ be matrices & $\lambda$ a scalar. The following formulas hold whenever the terms on one side are defined.

1. $A(B + C) = AB + AC$.
2. $(A + B)C = AC + BC$.
3. $(\lambda A)B = \lambda(AB) = A(\lambda B)$.

Proof. Easy ex similar to distributive law we proved for matrix-vector product.

Noncommutativity

Note that if $AB$ is defined, $BA$ may not be, and even if it is, usually we have $AB \neq BA$.

E.g. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Hence $(A + B)^2 = \ldots$
Distributive laws & noncommutativity

Distributive law

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Hence $(A + B)^2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$
Distributive laws & noncommutativity

### Distributive law

Let $A$, $B$, $C$ be matrices & $\lambda$ a scalar. The following formulas hold whenever the terms on one side are defined.

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**Formula**

$I A = A, B I = B$ whenever the products are defined.

**Proof.** Just multiply matrices. We'll check here

**Upshot** In particular, we see the linear function associated to $I$ is the identity map $x \mapsto x$. 
The transpose of an \( m \times n \)-matrix \( A \), is the \( n \times m \)-matrix \( A^T \) gotten by turning all the rows of \( A \) into columns (or equivalently, flipping the matrix about the row=column diagonal).

E.g. \[
\begin{pmatrix}
2 & 5 & 1 \\
1 & 3 & 2 \\
\end{pmatrix}
\]
\( \rightarrow \)

More formally, the entries of \( A^T \) are given by \[
A^T_{ij} = A_{ji}.
\]

Formulas

Let \( A, B \) be matrices & \( \lambda \) a scalar. The following hold when one side is defined.

1) \( (A^T)^T = A \)

2) \( (A + B)^T = A^T + B^T \)

3) \( (\lambda A)^T = \lambda A^T \)

4) \( (AB)^T = B^T A^T \)

Proof.

1) & 2) are easy and say the function \( A \mapsto A^T \) is linear. For 3) \[
\left[(B^T A^T)\right]_{ij} = \sum_l [B^T]_{il} [A^T]_{lj} = \sum_l [B]_{li} [A]_{jl} = \left[(\lambda A)^T\right]_{ij}.
\]

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Chapter 5: Matrices
The *transpose* of an $m \times n$-matrix $A$, is the $n \times m$-matrix $A^T$ gotten by turning all the rows of $A$ into columns (or equivalently, flipping the matrix about the row $i = \text{column } j$ diagonal).
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**Proof.** 1) & 2) are easy and say the function $A \mapsto A^T$ is linear. For 3)

\[
[(B^T A^T)]_{ij} = \sum_l [B^T]_{il} [A^T]_{lj} = \sum_l [B]_{li} [A]_{jl} =
\]
Miscellaneous tidbits involving transpose

Relation with dot product

Let \( a, b \in \mathbb{R}^n \). Then though \( ab \) is not defined we can define \( a^T b = (a_1 \ldots a_n)(b_1 \ldots b_n) = a_1b_1 + \ldots + a_nb_n = a \cdot b \).

Symmetric matrices

A square matrix is symmetric if \( A^T = A \) and anti-symmetric if \( A^T = -A \). (Why is square in the definition?)

E.g. \( bb^T \in M_{nn} \) is symmetric since

In fact if \( \|b\| = 1 \), then the linear function \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) associated to \( bb^T \) is projection onto \( b \) for it sends \( x \mapsto bb^T x \).
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In fact if \( |b| = 1 \), then the linear function \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) associated to \( bb^T \) is projection onto \( b \) for it sends \( x \mapsto bb^Tx \).
 Relation with dot product

Let \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \).  

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Matrix Inverse

To attempt matrix "division" we need the matrix inverse.

A matrix $W$ is an inverse for the matrix $A$ if $AW = WA = I$.

If such a matrix exists we say that $A$ is invertible.

A matrix can have at most one inverse, denoted $A^{-1}$.

Proof. If $W'$ is another inverse then $E.g. A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$.
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Proposition
Let $A \in \mathbb{M}_{mn}$ be an invertible matrix. Then for any $b \in \mathbb{R}^n$ the equation $Ax = b$ has $x = A^{-1}b$ as its unique solution.

Proof. $x = A^{-1}b$ is a solution since $A^{-1}b = b$.$\Rightarrow$ It is the only possible solution since $Ax = b$. 

Corollary
Any invertible matrix $A \in \mathbb{M}_{mn}$ is square.

Proof. The inverse $W = A^{-1}$ must be $n \times m$ for $AW$, $WA$ to be defined. $A^{-1}b = 0$ has a unique solution $\Rightarrow m$. 

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Inverses and solving $A\mathbf{x} = \mathbf{b}$

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Formulas involving matrix inverses

Let $A, B \in \mathbb{M}_{nn}$ be invertible.

1. $A^{-1}$ is invertible & $(A^{-1})^{-1} = A$.

2. $AB$ is invertible & $(AB)^{-1} = B^{-1}A^{-1}$.

3. $A^T$ is invertible & $(A^T)^{-1} = (A^{-1})^T$.

Proof. All easy e.g.
**Formulas involving matrix inverses**

<table>
<thead>
<tr>
<th>Formula</th>
</tr>
</thead>
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E.g. Simplify $\left(ABA^{-1}\right)^{-2}$.
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Computing inverses of matrices: example

Find the inverse of

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 3 & 2 \\
-1 & 1 & -1
\end{pmatrix}
\]

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Computing inverses of matrices: example

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Computing inverses of matrices: example

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The algorithm for inverting matrices

Let $A \in \mathbb{M}_{nn}$. To determine invertibility and invert $A$ if invertible we:

1. Form the augmented $n \times 2n$-matrix $(A | I)$.
2. Apply ERO's until we get row echelon form $(U | B)$.
3. If $U$ has non-leading columns, we stop as solns to $A x = 0$ are not unique so $A$ is not invertible.
4. If $U$ has no non-leading columns, then $A$ is invertible & we can apply EROS to transform $(U | B)$ to the form $(I | C)$ with $C = A^{-1}$.

Q: Is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 4 & 3 \end{pmatrix}$ invertible? If so, find its inverse.
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To see why the method works, we need to know

**Proposition**

1. There's an invertible matrix $E(i,j;c) \in M_{nn}$ such that $E(i,j;c)A$ is obtained from $A$ by the ERO $R_i = R_i + cR_j$.

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3. There's a matrix $E(i;c) \in M_{nn}$ such that $E(i;c)A$ is obtained from $A$ by the ERO $R_i = cR_i$. It's invertible if $c \neq 0$.

In other words, an ERO can be performed by left multiplication by a corresponding matrix.

**Proof.**

Just dream up the required matrices (see notes) e.g.
EROS and matrix multiplication

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Proof. Just dream up the required matrices (see notes) e.g.
Why our algorithm works

Hence associativity of matrix multiplication:

\[ \Rightarrow \]

\[ a \text{ sequence of EROs corresponding to matrices } E_1, \ldots, E_l, \text{ itself corresponds to left multn by } E_l \ldots E_2 E_1. \]

Thus if these EROs send \((A | I) \mapsto (I | C)\) we have:

\[ (I | C) = (E_l \ldots E_2 E_1 A | E_l \ldots E_2 E_1 I) = (E_l \ldots E_2 E_1 A | E_l \ldots E_2 E_1). \]

Hence \(C = E_l \ldots E_2 E_1 \) and \(CA = I\). But each \(E_i\) is invertible so the product \(C\) is also invertible and \(C = A^{-1}\).
Hence associativity of matrix muln $\implies$ a sequence of EROs corresponding to matrices $E_1, \ldots, E_l$, itself corresponds to left multn by $E_l \ldots E_2 E_1$. 

Thus if these EROs send $(A | I) \mapsto (I | C)$ we have $(I | C) = (E_l \ldots E_2 E_1 A | E_l \ldots E_2 E_1 I) = (E_l \ldots E_2 E_1 A | E_l \ldots E_2 E_1)$. 

Hence $C = E_l \ldots E_2 E_1$ and $CA = I$. But each $E_i$ is invertible so the product $C$ is too & $C = A^{-1}$. 

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$$

Hence $C = E_i \ldots E_2 E_1$ and $CA = I$. But each $E_i$ is invertible so the product $C$ is too & $C = A^{-1}$. 
Hence associativity of matrix muln \implies a sequence of EROs corresponding to matrices $E_1, \ldots, E_l$, itself corresponds to left multn by $E_l \ldots E_2 E_1$. Thus if these EROs send $(A|I) \mapsto (I|C)$ we have

$$(I|C) = (E_l \ldots E_2 E_1 A|E_l \ldots E_2 E_1 I) = (E_l \ldots E_2 E_1 A|E_l \ldots E_2 E_1)$$.
Hence associativity of matrix mulpn \implies a sequence of EROs corresponding to matrices \( E_1, \ldots, E_l \), itself corresponds to left mulpn by \( E_l \cdots E_2 E_1 \). Thus if these EROs send \((A|I) \mapsto (I|C)\) we have

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Hence \( C = E_l \cdots E_2 E_1 \) and \( CA = I \). But each \( E_j \) is invertible so the product \( C \) is too & \( C = A^{-1} \).
Invertibility for square matrices

Theorem

Suppose that $A \in \mathbb{M}_{nn}$ has row-echelon form $U$. Then the following are equivalent:

1. $A$ is invertible.
2. $U$ has no zero rows.
3. $U$ has no nonleading columns.
4. $Ax = 0$ has a unique solution $x = 0$.
5. For each $b \in \mathbb{R}^n$, $Ax = b$ has a unique solution.

Proof

This follows from our algorithm for inversion & the fact that in a square matrix if every row is a leading row, then every column is a leading column.

Corollary

Suppose that $A \in \mathbb{M}_{nn}$. Then $XA = I \iff AX = I \iff X = A^{-1}$.

Proof.

Note $AX$ or $XA$ square $\Rightarrow X \in \mathbb{M}_{nn}$, so by symmetry, suff show $AX = I \Rightarrow X$ is invertible.

But $Xx = 0 \Rightarrow x = 0$ so $AX = 0 \Rightarrow X = A^{-1}$ so we're done by the thm.
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Inverse of 2 × 2-matrices

Recall the 2 × 2-determinant

\[
\begin{vmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{vmatrix}
= \det(A) = a_{11}a_{22} - a_{12}a_{21}.
\]

Inverse formula

Let

\[A = (a_{ij}) \in M_{22}.\]

Then \(A\) is invertible if \(\det(A) \neq 0\) in which case

\[A^{-1} = \frac{1}{\det(A)} \left( a_{22} - a_{12} - a_{21}a_{11} \right).\]

Proof.

Just multiply

Remark

We'll generalise the relationship between determinants and invertibility later.

The formula in this case is best appreciated if you understand the linear mapping \(\mathbb{R}^2 \to \mathbb{R}^2\) associated to \(A\).
Inverse of $2 \times 2$-matrices

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Chapter 5: Matrices

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Chapter 5: Matrices

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Daniel Chan (UNSW)
We define the determinant of $A \in M_{nn}$, using induction on $n$.

We've defined it for $n = 2$ (and 3), and suppose it's defined for $(n-1) \times (n-1)$-matrices. First definition:

For $1 \leq i, j \leq n$, the $(row\ i, \ column\ j)$ minor of $A$, denoted $|A|_{ij}$, is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row $i$ and column $j$ from $A$.

E.g. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 4 & -2 & 5 \end{pmatrix}$.

$|A|_{21} = $
Minors of a square matrix

We define the *determinant* of $A \in M_{nn}$, using induction on $n$. 

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$|A|_{21} = \text{Defn}$

For a $1 \times 1$-matrix $(a)$ we define its determinant to be $\det(a) = a$. 

There are many definitions of the determinant. To keep the exposition elementary, we'll use a generalisation of the one for $3 \times 3$-determinants.
Minors of a square matrix

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**Example**

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Determinant

This definition is via expanding along the first row.

Definition

Let $A \in \mathbb{M}_{nn}$ with $n \geq 2$. Then the determinant of $A$ is

$$\det(A) = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}| + \cdots + (-1)^{n+1}|A_{1n}|.$$

E.g.

Let $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ -1 & 1 & 5 & 0 \\ 7 & -2 & 3 & 1 \end{pmatrix}$.

Then $\det(A) = \text{...}$
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Determinant of lower triangular matrices

We say $A = (a_{ij})$ is lower triangular, if $a_{ij} = 0$ whenever $i < j$.

The computation above shows more generally the following proposition:

If $A = (a_{ij})$ is lower triangular, then $\det(A)$ is just the product of the diagonal elements.

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

In particular, $\det(I) = 1$. 

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Determinant of lower triangular matrices

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In particular, $\det(I) = 1$. 

Determinant of lower triangular matrices

We say \( A = (a_{ij}) \) is *lower triangular*, if \( a_{ij} = 0 \) whenever \( i < j \).

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The geometric meaning of the determinant

Unfortunately in this course, we will not be able to talk about the determinant in depth. For a thorough treatment, see my MATH2601 lecture notes, lectures 5, 6.

Just as the \(2 \times 2\)-determinant gives the area of parallelograms, and \(3 \times 3\)-determinants volumes of parallelopipeds, the \(n \times n\)-determinant gives the volume of \(n\)-dimensional parallelopipeds (unfortunately we can't prove this).

In particular, the linear mapping associated \(\mathbb{R}^n \rightarrow \mathbb{R}^n\) to \(A\) expands volumes by \(\det(A)\).

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Determinant of the Transpose

Let $A = (a_{ij}) \in M_{nn}$. Then

**Proposition** $\det(A) = \det(A^T)$.

**Proof.**

Hard (& omitted) with our defn except for $2 \times 2$.

Since the transpose swaps columns with rows, we can compute $\det(A)$ by expanding along the first column as in the formula below

$$\det(A) = a_{11} |A_{11}| - a_{21} |A_{21}| + \cdots + (-1)^{n+1} a_{n1} |A_{n1}|.$$  

In particular, if $A$ is upper triangular in the sense that $a_{ij} = 0$ whenever $i > j$, then $\det(A)$ is the product of the diagonal entries. E.g.
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E.g. Daniel Chan (UNSW) Chapter 5: Matrices Semester 1 2015 30 / 35
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**E.g.**
Determinant of products

Theorem
\[ \det(AB) = \det(A) \det(B) \]

Note that \( A, B \) square and \( AB \) defined \( \Rightarrow \) \( A, B \) have the same size.

Proof. Hard, see my MATH2601 notes.

E.g. Find the determinant of \( A \) where
\[
A = \begin{pmatrix}
1 & 0 & 0 & 42 \\
4 & e^{\pi} & -\sqrt{2} & 1 \\
2 & 0 & \int_0^1 \frac{1}{\sqrt{(1-x^2)(2-x^2)}} \, dx
\end{pmatrix}
\]

Corollary
If \( A \) is invertible, then \( \det(A) \neq 0 \) & \( \det(A^{-1}) = \det(A)^{-1} \).

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**Proof.** Hard, see my MATH2601 notes.

**E.g.** Find the determinant of \(A^4\) where \(A = \begin{pmatrix}
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**Proof.**
Below we see how performing an ERO to get from $A \xrightarrow{\text{ERO}} B$ changes the determinant.

**Proposition 1**
If $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $\det(B) = -\det(A)$. i.e. swapping two rows of a matrix negates the determinant.

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If $A \xrightarrow{R_i = cR_i} B$, then $\det(B) = c\det(A)$. i.e, multiplying a row by $c$ multiplies the determinant by $c$.

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If $A \xrightarrow{R_i = R_i + cR_j} B$, then $\det(B) = \det(A)$. i.e, In particular, if two rows of $A$ are the same, or $A$ has a row of 0s, then $\det(A) = 0$.

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Recall that $B = EA$ for some square matrix $E$. We need only check $\det(E) = -1$, $c$, $1$ in cases 1), 2), 3) resp. e.g.

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Daniel Chan (UNSW)
Warning: We don't compute determinants using the definition except for small or special matrices! In practice, a) we apply EROs to reduce $A$ to row-echelon form $U$, b) record how we've changed the determinant (see previous slide), and c) note $U$ is square & row-echelon $\Rightarrow$ upper triangular so has readily computable determinant.

E.g. Find the determinant of $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 1 & 1 & 2 \\ -1 & 2 & 0 & 1 \\ 2 & 0 & 3 \\ -1 \end{bmatrix}$.
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Then $\det(A) = \pm \det(U)$.

For this $U$, it has all leading columns iff all diagonal entries are non-zero iff $\det(U) \neq 0$.

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**Theorem**

A square matrix $A$ is invertible iff $\det(A) \neq 0$.

**Remark**

The formula for inverting $2 \times 2$-matrices generalises (see Cramer's rule in my MATH2601 notes).

It has $\det(A)$ in the denominator for $A^{-1}$ and an otherwise well-defined numerator.
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- We can transform $A$ into upper echelon form $U$ by using ERO’s of form $R_i \leftrightarrow R_j$ & $R_i = R_i + cR_j$ only. Then $\det(A) = \pm \det(U)$.
- For this $U$, it has all leading columns iff all diagonal entries are non-zero iff $\det(U) \neq 0$.

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A square matrix $A$ is invertible iff $\det(A) \neq 0$. 

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The formula for inverting $2 \times 2$-matrices generalises (see Cramer’s rule in my MATH2601 notes). It has $\det(A)$ in the denominator for $A^{-1}$ and an otherwise well-defined numerator.
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