Chapter 4: Linear Equations

Daniel Chan

UNSW

Semester 1 2018
An Ancient Chinese Problem

The following maths problem was taken from the Chinese maths text "Nine Chapters on the Mathematical Arts" ca. 200CE

Problem

One pint of good wine costs 50 gold pieces, one pint of bad only 10. If two pints are bought for 30 gold pieces, how much of each wine is bought?

The Chinese knew this is a simple algebra problem. If \( x \), \( y \) are the amount (in pints) of good and bad wine bought then

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\begin{align*}
x + y &= 2 \quad (1) \\
50x + 10y &= 30 \quad (2)
\end{align*}
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We review the solution, as we wish to generalise this later.

Key

Eliminate variables.

This can be done in lots of ways but the method we'll generalise is

$$\text{Subtract } 50 \times \text{eqn (1)} \text{ from eqn (2) to get:}$$

$$-40y = -70 \implies y = \frac{7}{4}$$

Now obtain $x$ by using this and eqn (1) (or (2))

$$x = 2 - y = 1\frac{3}{4}.$$
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Linear equations

A linear equation in \( n \) unknowns \( x_1, \ldots, x_n \) is an equation of the form

\[
a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b
\]

where \( a_1, \ldots, a_n, b \) are given real numbers.

E.g.

The name linear comes from

We are interested in solving several of these simultaneously as in the ancient Chinese problem.

A system of \( m \) linear equations in \( n \) variables is

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\begin{align*}
  a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
  a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\
  \vdots \ & \quad \vdots \\
  a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m
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A linear equation in $n$ unknowns $x_1, \ldots, x_n$ is an equation of the form

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Point normal form for lines in $\mathbb{R}^2$

Just as there are point normal forms for planes in $\mathbb{R}^3$, there are point normal forms for lines in $\mathbb{R}^2$ (can you see the general pattern??)

Let $L \subset \mathbb{R}^2$ be the line in $\mathbb{R}^2$ passing through the point $a$ with normal vector $n = (n_1, n_2)$.

$L$ is the set of points $x$ satisfying the point normal form $n \cdot (x - a) = 0 \iff n_1 x_1 + n_2 x_2 = n \cdot x = n \cdot a$. 

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\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0 \iff n_1 x_1 + n_2 x_2 = \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{a}.
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Intersecting 2 lines in $\mathbb{R}^2$

Consider solving a system of 2 eqns in 2 unknowns $x, y$:

1. $a_{11}x + a_{12}y = b_1$ (3)
2. $a_{21}x + a_{22}y = b_2$ (4)

Plotting solutions in the $xy$-plane allows us to understand the nature of solutions better.

Assume both eqns are non-zero (otherwise ??) so eqns (3), (4) define lines $L_1, L_2$ and $L_1 \cap L_2$ represents the simultaneous soln.

On geometric grounds, we see there are ?? possibilities

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2 linear equations in 2 unknowns: algebraic version

L_1 = L_2 \iff \text{eqn (3) and (4) are scalar multiples of each other. Simultaneous soln same as the soln to one of them.}

L_1 \parallel L_2 \text{ but not equal. This means that the normals are parallel so } \left( a_{11} a_{12} \right) = r \left( a_{21} a_{22} \right), \text{ for some } r \in \mathbb{R}.

\text{Then (3) } - r \times (4) \text{ gives } 0 = b_1 - r b_2 \neq 0 \text{ WHY? so there are no solns.}

\left( a_{11} a_{12} \right), \left( a_{21} a_{22} \right) \text{ are not parallel. Our procedure gives a unique soln. This is easy to see from any e.g. such as in the ancient Chinese problem.}
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Intersecting 3 planes in $\mathbb{R}^3$

Here are some possibilities for intersecting 3 planes $P_1, P_2, P_3 \subset \mathbb{R}^3$.

- The planes intersect in a point.
- There is no intersection as the three are parallel and distinct.
- $P_1, P_2$ intersect in a line which is parallel to $P_3$. There's no intersection.
- $P_1 = P_2$ intersect $P_3$ in a line so the three intersect in a line.

etc

There are heaps more cases, which makes it hard to analyse.

We need make our method of solving simultaneous solns and understanding the set of solns systematic.

This procedure, described in the "Nine Chapters" and re-discovered by Gauss is called Gaussian elimination.

Aside Gauss developed it to solve a system of 6 linear eqns in 6 unknowns and hence compute the orbit of the dwarf planet Ceres.

Look up Stigler's law on Wikipedia.
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There are heaps more cases, which makes it hard to analyse. We need make systematic, our method of solving simultaneous solns and understanding the set of solns.

This procedure, described in the “Nine Chapters” and re-discovered by Gauss is called Gaussssian elimination.

Aside Gauss developed it to solve a system of 6 linear eqns in 6 unknowns and hence compute the orbit of the dwarf planet Ceres. Look up Stigler’s law on Wikipedia.
Augmented matrix form

We first abbreviate our system of $m$ linear equations in $n$ variables:

$$
\begin{align*}
&\begin{bmatrix}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
  \vdots & \vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{bmatrix}
\end{align*}
$$

Here the notation omits the unknowns $x_1, \ldots, x_n$ which must be understood from the context (or irrelevant).
Augmented matrix form

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    \vdots & \quad & \vdots & \quad & \vdots \\
    a_{m1}x_1 & \quad + \quad a_{m2}x_2 & \quad + \quad \cdots & \quad + \quad a_{mn}x_n & \quad = \quad b_m
\end{align*}
\]

to **augmented matrix form**:

\[
\begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\
    a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\
    \vdots & \vdots & \ddots & \vdots & | & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn} & | & b_m
\end{pmatrix}
\]
Augmented matrix form

We first abbreviate our system of $m$ linear equations in $n$ variables:

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\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
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    &\vdots \quad \quad \vdots \quad \quad \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

to augmented matrix form:

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\begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
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    \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{pmatrix}
\]

Here the notation omits the unknowns $x_1, \ldots, x_n$ which must be understood from the context (or irrelevant).
Example: augmented matrix form

Write the augmented matrix form for the system:
\[
\begin{align*}
6x_1 - 2x_2 + 5x_3 &= 7 \\
3x_1 + x_2 + 2x_3 &= 0
\end{align*}
\]

Solution.

Write the system of equations with augmented matrix form:
\[
\begin{pmatrix}
2 & 1 & 1 \\
-3 & 3 & 5 & 0 & 7
\end{pmatrix}
\]
Example: augmented matrix form

E.g. Write the augmented matrix form for the system

\[
\begin{align*}
6x_1 & - 2x_2 + 5x_3 = 7 \\
3x_1 & + x_2 + 2x_3 = 0
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**E.g.** Write the augmented matrix form for the system

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3x_1 + x_2 + 2x_3 &= 0
\end{align*}
\]

**Solution.**

**E.g** Write the system of equations with augmented matrix form

\[
\begin{pmatrix}
2 & 1 & 1 & \mid & -3 \\
3 & 5 & 0 & \mid & 7
\end{pmatrix}
\]
Matrices

A matrix is just a rectangular array of numbers, like
\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 7 & 5 \\
\end{pmatrix}
\]
We would say that this is a $2 \times 3$ matrix as it has 2 rows and 3 columns.

An $m \times 1$ matrix just looks like a vector
\[
\mathbf{b} = \begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m \\
\end{pmatrix} \in \mathbb{R}^m
\]
The augmented matrix consists of the coefficient matrix (on the left of the line) augmented by the right-hand-side vector. We can write
\[
\begin{pmatrix}
2 & 1 & 1 \\
3 & 5 & 0 \\
\end{pmatrix}
\]
as
\[
(A | b)
\]
where
\[
A = \begin{pmatrix}
2 & 1 & 1 \\
3 & 5 & 0 \\
\end{pmatrix}
\]
and
\[
b = \begin{pmatrix}
-3 \\
7 \\
\end{pmatrix}
\]
A *matrix* is just a rectangular array of numbers, like

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 7 & 5
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\mathbf{b}_2 \\
\vdots \\
\mathbf{b}_m
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\]

The augmented matrix consists of the coefficient matrix (on the left of the line) augmented by the right-hand-side vector. We can write

\[
\begin{pmatrix}
2 & 1 & 1 \\
-3 & 3 & 5 & 0
\end{pmatrix}
\]

as

\[
\begin{pmatrix}
A \\
b
\end{pmatrix}
\]

where $A = \begin{pmatrix} 2 & 1 & 1 \\
-3 & 3 & 5 & 0\end{pmatrix}$ and $b = \begin{pmatrix} -3 \\
7 \end{pmatrix}$. 
Matrices

A *matrix* is just a rectangular array of numbers, like

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A *matrix* is just a rectangular array of numbers, like
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We would say that this is a $2 \times 3$ matrix as it has 2 rows and 3 columns.

An $m \times 1$ matrix just looks like a vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$. 
Matrices

A *matrix* is just a rectangular array of numbers, like

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 7 & 5
\end{pmatrix}.
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We would say that this is a $2 \times 3$ matrix as it has 2 rows and 3 columns.

An $m \times 1$ matrix just looks like a vector $\mathbf{b} = 
\begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{pmatrix} \in \mathbb{R}^m$

The augmented matrix consists of the *coefficient matrix* (on the left of the line) augmented by the right-hand-side vector. We can write
A matrix is just a rectangular array of numbers, like

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 7 & 5
\end{pmatrix}.
\]

We would say that this is a $2 \times 3$ matrix as it has 2 rows and 3 columns.

An $m \times 1$ matrix just looks like a vector $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$

The augmented matrix consists of the coefficient matrix (on the left of the line) augmented by the right-hand-side vector. We can write\[
\begin{pmatrix}
2 & 1 & 1 \\
3 & 5 & 0
\end{pmatrix}
\begin{pmatrix}
-3 \\
7
\end{pmatrix}
\]
as $(A|b)$ where $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 5 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$.
Simultaneous equations as a single vector equation

Any system of (not necessarily linear) simultaneous eqns can be written as a single vector eqn. The above eqn \((A|b)\) can be re-written as

\[
\begin{pmatrix}
-3 & 7 \\
2 & x_1 \\
3 & x_1 + 5 & x_2 \\
& & 3 & x_1 + 1 & x_2 + 1 & x_3 \\
\end{pmatrix} =
\begin{pmatrix}
2 & x_1 \\
\end{pmatrix}
\]

or perhaps as

\[
x_1 a_1 + x_2 a_2 + x_3 a_3 = b.
\]

where \(a_1, a_2, a_3\) are the columns of \(A\). That is, solving this system is equivalent to asking:

How can we write \(b\) as a linear combination of the vectors \(a_1, a_2\) and \(a_3\)?
Simultaneous equations as a single vector equation

Any system of (not necessarily linear) simultaneous eqns can be written as a single vector eqn.

$$\begin{pmatrix} -3 & 7 \end{pmatrix} = \begin{pmatrix} 2 \end{pmatrix} x_1 + x_2 + x_3$$

or perhaps as

$$x_1 a_1 + x_2 a_2 + x_3 a_3 = b.$$
Simultaneous equations as a single vector equation

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The above eqn \((A|b)\) can be re-written as

\[
\begin{pmatrix}
-3 & 7 & \\
5 & & \\
3 & x_1 & + & 5 & x_2 & + & x_3 & = & 2
\end{pmatrix}
\]

or perhaps as

\[
x_1 a_1 + x_2 a_2 + x_3 a_3 = b.
\]

where \(a_1, a_2, a_3\) are the columns of \(A\).
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Simultaneous equations as a single vector equation

Any system of (not necessarily linear) simultaneous eqns can be written as a single vector eqn.

The above eqn \((A|b)\) can be re-written as

\[
\begin{pmatrix}
-3 \\
7
\end{pmatrix} = \begin{pmatrix}
2x_1 + x_2 + x_3 \\
3x_1 + 5x_2
\end{pmatrix} = x_1 \begin{pmatrix}
2 \\
3
\end{pmatrix} + x_2 \begin{pmatrix}
1 \\
5
\end{pmatrix} + x_3 \begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

or perhaps as

\[
x_1 a_1 + x_2 a_2 + x_3 a_3 = b
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7
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2x_1 + x_2 + x_3 \\
3x_1 + 5x_2
\end{pmatrix}
= x_1 \begin{pmatrix}
2 \\
3
\end{pmatrix} + x_2 \begin{pmatrix}
1 \\
5
\end{pmatrix} + x_3 \begin{pmatrix}
1 \\
0
\end{pmatrix}
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or perhaps as

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2x_1 + x_2 + x_3 \\
3x_1 + 5x_2
\end{pmatrix}
= x_1 \begin{pmatrix}
2 \\
3
\end{pmatrix} + x_2 \begin{pmatrix}
1 \\
5
\end{pmatrix} + x_3 \begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

or perhaps as

\[x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3 = \mathbf{b}.\]

where \(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\) are the columns of \(A\).
Any system of (not necessarily linear) simultaneous eqns can be written as a single vector eqn.

The above eqn \((A|b)\) can be re-written as

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\begin{pmatrix}
-3 \\ 7
\end{pmatrix} = \begin{pmatrix}
2x_1 + x_2 + x_3 \\ 3x_1 + 5x_2
\end{pmatrix} = x_1 \begin{pmatrix}
2 \\ 3
\end{pmatrix} + x_2 \begin{pmatrix}
1 \\ 5
\end{pmatrix} + x_3 \begin{pmatrix}
1 \\ 0
\end{pmatrix}
\]

or perhaps as

\[x_1 a_1 + x_2 a_2 + x_3 a_3 = b.\]

where \(a_1, a_2, a_3\) are the columns of \(A\).

That is, solving this system is equivalent to asking:

*How can we write \(b\) as a linear combination of the vectors \(a_1, a_2\) and \(a_3\)?*
Examples: different forms for linear equations

Q
The following vector equation is equivalent to which system of linear equations
and which augmented matrix?

\[ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 9 \end{pmatrix} \]

Q
Write a vector equation equivalent to the following augmented matrix.

\[ \begin{pmatrix} 1 & 0 & -2 \\ 2 & e^{2\pi} & \end{pmatrix} \]
The following vector equation is equivalent to which system of linear equations and which augmented matrix?

\[
x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}
\]
Q The following vector equation is equivalent to which system of linear equations and which augmented matrix?

\[ x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \]

Q Write a vector equation equivalent to the following augmented matrix.

\[
\begin{pmatrix}
1 & 0 & -2 \\
2 & e^2 & \pi
\end{pmatrix}
\]
Equivalent systems of linear equations

We say a system of linear equations is consistent if it has at least one solution and inconsistent otherwise.

E.g.,
\[ x + y = 2, \]
\[ 2x + 2y = 5 \]
is consistent.

**Definition**
Two systems of linear equations \((A|b), (A'|b')\) are said to be equivalent if they have exactly the same set of solutions.

The following define equivalent systems of linear equations:

\[ \begin{align*}
  x + y &= 2 \\
  2x + y &= 2 
\end{align*} \] (5)

\[ \begin{align*}
  2x + 3y &= 5 \\
  3x + 4y &= 7 
\end{align*} \] (6)

Why?
Note that the 2nd system is gotten from the 1st by adding eqn (5)LHS to eqn (6)LHS. Similarly, the 1st system...
Equivalent systems of linear equations

We say a system of linear eqns is *consistent* if it has at least one solution and *inconsistent* otherwise.

E.g.

\[ \begin{align*}
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The systems are equivalent because
Equivalent systems of linear equations

We say a system of linear eqns is *consistent* if it has at least one solution and *inconsistent* otherwise.

*E.g.* \( x + y = 2, 2x + 2y = 5 \) is
Equivalent systems of linear equations

We say a system of linear eqns is *consistent* if it has at least one solution and *inconsistent* otherwise.

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The following define equivalent systems of linear equations

\[
\begin{align*}
\text{(5)} & \quad x + y = 2 \\
& \quad 2x + 3y = 5 \\
& \quad 3x + 4y = 7 \\
\text{(6)} & \quad x + y = 2 \\
& \quad 2x + 3y = 5 \\
& \quad 3x + 4y = 7
\end{align*}
\]

**Why?** Note that the 2nd system is gotten from the 1st by adding eqn (5)LHS to eqn (6)LHS.
Equivalent systems of linear equations

We say a system of linear eqns is \textit{consistent} if it has at least one solution and \textit{inconsistent} otherwise.

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\textbf{Definition}

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**Why?** Note that the 2nd system is gotten from the 1st by adding eqn (5)LHS to eqn (6)LHS. Similarly, the 1st system
The systems are equivalent because
Elementary row operations

To solve \((A | b)\), we pass to an equivalent system of eqns where one of the eqns has only 1 unknown. We can solve this and then by induction on the number of variables, should be done.

How do you pass to an equivalent system of linear equations? Any of the following procedures will yield an equivalent system:

Elementary row operations

- Swap the order in which you write the equations!
- Multiply any equation by a non-zero constant. (Why non-zero?)
- Add a multiple of one equation to another equation. For example, you can replace equation (5) with equation (5) + 7 \times equation (6).

These are called the elementary row operations.

This is all easier to do if we use the augmented matrix form! We just replace the word eqn above with row of the augmented matrix.
Elementary row operations

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Elementary row operations

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Q How do you pass to an equivalent system of linear equations?
Elementary row operations

To solve \((A\mid b)\), we pass to an equivalent system of eqns where one of the eqns has only 1 unknown. We can solve this and then by induction on the number of variables, should be done.

Q How do you pass to an equivalent system of linear equations?
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  equation (5) with
  \[ \text{equation (5) + 7 \times \text{equation (6)}.} \]
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equation (5) + 7 \times \text{equation (6)}.

These are called the **elementary row operations**.

This is all easier to do if we use the augmented matrix form! We just replace the word eqn above with row of the augmented matrix.
Notation for EROs

We record the elementary row operation on an arrow between the old system and the new. In each case below, the system on the right is equivalent to the left.

\[
\begin{pmatrix}
0 & 2 & 3 \\
1 & 3 & 1 \\
2 & 4 & 7
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 2 & 3 \\
−1 & 3 & 1 \\
2 & 4 & 7
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
−1 & 4 & 1 \\
2 & 10 & 7
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3 \\
0 & 4 & 2 \\
2 & 10 & 7
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 \\
0 & 4 & 2 \\
0 & 0 & 4
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & 0 & 3 & 6 \\
0 & 0 & 0 & 4
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
Notation for EROs

We record the elementary row operation on an arrow between the old system and the new. In each case below, the system on the right is equivalent to the left.
Notation for EROs

We record the elementary row operation on an arrow between the old system and the new. In each case below, the system on the right is equivalent to the left.

\[
\begin{pmatrix}
0 & 2 & 3 & 5 \\
-1 & 3 & 1 & 6 \\
2 & 4 & 7 & 8 \\
\end{pmatrix}
\xrightarrow{R_2 \leftrightarrow R_1}
\begin{pmatrix}
-1 & 3 & 1 & 6 \\
0 & 2 & 3 & 5 \\
2 & 4 & 7 & 8 \\
\end{pmatrix}.
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 5 \\
-1 & -4 & 1 & 6 \\
2 & 10 & 7 & 8 \\
\end{pmatrix}
\xrightarrow{R_2 = R_2 + R_1}
\begin{pmatrix}
1 & 2 & 3 & 5 \\
0 & -2 & 4 & 11 \\
2 & 10 & 7 & 8 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 8 \\
0 & 0 & 3 & 6 & 9 \\
0 & 0 & 0 & 4 & 8 \\
\end{pmatrix}
\xrightarrow{R_2 = \frac{1}{3}R_2}
\begin{pmatrix}
1 & 2 & 3 & 4 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 2 \\
\end{pmatrix}.\]
We wish to apply a series of EROs until we arrive at a system of linear eqns we can solve!

Which systems are these?

Ones like the bottom right one on the previous slide.

Remember, that augmented matrix is just saying:

\[
\begin{align*}
    x_1 + 2x_2 + 3x_3 + 4x_4 &= 8 \\
    x_3 + 2x_4 &= 3 \\
    x_4 &= 2
\end{align*}
\]

Starting from the bottom, you can first solve \( x_4 \), and then moving up solve \( x_3 \) and then (see later) \( x_2 \), \( x_1 \).

Such a system is said to be in row-echelon form.

Solving it by starting from the bottom eqn and progressively moving up is called back substitution.
We wish to apply a series of EROs until we arrive at a system of linear eqns we can solve!
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Q Which systems are these?
Row echelon form: motivation

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    x_3 & \quad + \quad 2x_4 & \quad = \quad 3 \\
    x_4 & \quad = \quad 2
\end{align*}
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    x_3 & + 2x_4 &= 3 \\
    x_4 &= 2
\end{align*}
\]

Starting from the bottom, you can first solve \( x_4 \), and then moving up solve \( x_3 \) and then (see later) \( x_2, x_1 \).

Such a system is said to be in row-echelon form. Solving it by starting from the bottom eqn and progressively moving up is called back substitution.
Row echelon form

In any matrix:
1. A leading row is one which is not all zeros,
2. the leading entry in a leading row is the first (i.e. leftmost) non-zero entry,
3. a leading column is a column which contains the leading entry for some row.

Row echelon form
A matrix is said to be in row-echelon form if
1. all leading rows are above all non-leading rows (so any all-zero rows are at the bottom of the matrix), and
2. in every leading row, the leading entry is further to the right than the leading entry in any row higher up in the matrix.

If you remove the zeros, row echelon form gives an upside-down staircase shape.

Reduced row echelon form
A matrix is in reduced row echelon form if 1) it is in row echelon form, 2) all leading entries are 1 and, 3) all entries above leading entries are 0.
Row echelon form

In any matrix

1. a *leading row* is one which is not all zeros,
Row echelon form

In any matrix

1. a *leading row* is one which is not all zeros,
2. the *leading entry* in a leading row is the first (i.e. leftmost) non-zero entry,
Row echelon form

In any matrix

1. a **leading row** is one which is not all zeros,
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**Row echelon form**

A matrix is said to be in **row-echelon form** if
Row echelon form

In any matrix

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2. the *leading entry* in a leading row is the first (i.e. leftmost) non-zero entry,
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Row echelon form

A matrix is said to be in *row-echelon form* if

1. all leading rows are above all non-leading rows (so any all-zero rows are at the bottom of the matrix), and
Row echelon form

In any matrix

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Row echelon form

A matrix is said to be in **row-echelon form** if

1. all leading rows are above all non-leading rows (so any all-zero rows are at the bottom of the matrix), and
2. in every leading row, the leading entry is further to the right than the leading entry in any row higher up in the matrix.

If you remove the zeros, row echelon form gives an upside-down staircase shape.

A matrix is in **reduced row echelon form** if it 1) is in row echelon form, 2) all leading entries are 1 and, 3) all entries above leading entries are 0.
Examples: leading columns, row echelon form

E.g. Check the following is in row echelon form and solve:

\[
\begin{pmatrix}
1 & 2 \\
3 & 0 & 2
\end{pmatrix}
\]

E.g.

\[
\begin{pmatrix}
1 & 2 & 3 \\
-1 & 0 & 0 & 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & \frac{\pi}{2} \\
0 & 0 & e^0
\end{pmatrix}
\]
Examples: leading columns, row echelon form

E.g. Check the following is in row echelon form and solve

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 2 & 4
\end{pmatrix}
\]
Examples: leading columns, row echelon form

E.g. Check the following is in row echelon form and solve

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 2 & 4 \\
\end{pmatrix}
\]

E.g.

\[
\begin{pmatrix}
1 & 2 & 3 & -1 \\
0 & 0 & 2 & \pi \\
0 & 0 & e & 0 \\
\end{pmatrix}
\]
Back-substitution

We can solve systems of linear eqns if they are in row echelon form as in the following e.g.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\((R_1)\) \((R_2)\) \((R_3)\)

From \((R_3)\), \(x_4 = 2\).

Back substituting this into \((R_2)\) you get \(x_3 = \) .

The convention is that you set the variable for any non-leading column to be an arbitrary parameter.

So, let \(x_2 = \lambda\).

Then \(x_1 = 8 - 2\lambda - 3 \times (-1) - 4 \times 2 = 3 - 2\lambda\).
Back-substitution

We can solve systems of linear eqns if they are in row echelon form as in the following e.g.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

From (R₃), \(x₄ = 2\).

Back substituting this into (R₂) you get \(x₃ = \) .

The convention is that you set the variable for any non-leading column to be an arbitrary parameter.

So, let \(x₂ = \lambda\).

Then \(x₁ = 8 - 2\lambda - 3 \times (-1) - 4 \times 2 = 3 - 2\lambda\).
Back-substitution

We can solve systems of linear eqns if they are in row echelon form as in the following e.g.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 2 \\
\end{pmatrix}
\]

\((R1)\)
\((R2)\)
\((R3)\)

From \((R3)\), \(x_4 = 2\).

Back substituting this into \((R2)\) you get \(x_3 = \cdot\).

The convention is that you set the variable for any non-leading column to be an arbitrary parameter.

So, let \(x_2 = \lambda\).

Then
\[
x_1 = 8 - 2\lambda - 3 \times (-1) - 4 \times 2 = 3 - 2\lambda.
\]
We can solve systems of linear eqns if they are in row echelon form as in the following e.g.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 2 \\
\end{pmatrix}
\]

\((R1)\) \hspace{1cm} \((R2)\) \hspace{1cm} \((R3)\)

From \((R3)\), \(x_4 = 2\).
Back-substitution

We can solve systems of linear eqns if they are in row echelon form as in the following e.g.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 2
\end{pmatrix}
\quad (R1)
\]

\[
\begin{pmatrix}
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 2
\end{pmatrix}
\quad (R2)
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 & 2
\end{pmatrix}
\quad (R3)
\]

From \((R3)\), \(x_4 = 2\). Back substituting this into \((R2)\) you get \(x_3 = \ldots\).
Back-substitution

We can solve systems of linear eqns if they are in row echelon form as in the following e.g.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 2
\end{pmatrix} \quad (R1) \\
\begin{pmatrix}
1 & 2 & 3 & 4 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 2
\end{pmatrix} \quad (R2) \\
\begin{pmatrix}
1 & 2 & 3 & 4 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 2
\end{pmatrix} \quad (R3)
\]

From \((R3)\), \(x_4 = 2\). Back substituting this into \((R2)\) you get \(x_3 = \) .

The convention is that you set the variable for any non-leading column to be an arbitrary parameter.
We can solve systems of linear eqns if they are in row echelon form as in the following e.g.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & | & 8 \\
0 & 0 & 1 & 2 & | & 3 \\
0 & 0 & 0 & 1 & | & 2 \\
\end{pmatrix} \quad (R1)
\]

\[
\begin{pmatrix}
0 & 0 & 1 & 2 & | & 3 \\
0 & 0 & 0 & 1 & | & 2 \\
\end{pmatrix} \quad (R2)
\]

From \((R3)\), \(x_4 = 2\). Back substituting this into \((R2)\) you get \(x_3 = \ldots\).

The convention is that you set the variable for any non-leading column to be an arbitrary parameter.

So, let \(x_2 = \lambda\).
We can solve systems of linear eqns if they are in row echelon form as in the following e.g.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 2 \\
\end{pmatrix} \quad (R1)
\]

\[
\begin{pmatrix}
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} \quad (R2)
\]

From \((R3)\), \(x_4 = 2\). Back substituting this into \((R2)\) you get \(x_3 = \) .

The convention is that you set the variable for any non-leading column to be an arbitrary parameter.

So, let \(x_2 = \lambda\). Then

\[
x_1 = 8 - 2\lambda - 3 \times (-1) - 4 \times 2 = 3 - 2\lambda.
\]
Vector form of the solution

It is instructive to write the solution in vector form:

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} =
\begin{bmatrix}
  3 \\
  -2 \\
  \lambda \\
  \lambda - 1
\end{bmatrix}
\]

This shows the solution set is a line in \(\mathbb{R}^4\) in the direction \(\lambda\).

The argument shows that you will have as many parameters \(\lambda_1, \ldots, \lambda_j\) as you have non-leading columns.

Geometrically, the solution set is \(j\)-dimensional.

We usually expect you to write your answer in vector form as above.
It is instructive to write the soln in vector form

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} =
\begin{bmatrix}
    3 \\
    -2 \\
    \lambda \\
    \lambda
\end{bmatrix}
\]

This shows the solution set is a line in \( \mathbb{R}^4 \) in the direction \( (1, -1, 2, 2)^T \).

This argument shows that you will have as many parameters \( \lambda_1, \ldots, \lambda_j \) as you have non-leading columns.

Geometrically, the solution set is \( j \)-dimensional.
Vector form of the solution

It is instructive to write the soln in vector form

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{pmatrix}
= 
\begin{pmatrix}
  3 - 2\lambda \\
  \lambda \\
  -1 \\
  2
\end{pmatrix}
= 
\begin{pmatrix}
  3 - 2\lambda \\
  \lambda \\
  -1 \\
  2
\end{pmatrix}
\]

This shows the solution set is a line in \( \mathbb{R}^4 \) in the direction.

**Upshot**

This argument shows that you will have as many parameters \( \lambda_1, \ldots, \lambda_j \) as you have non-leading columns.

Geometrically, the solution set is \( j \)-dimensional.

We usually expect you to write your answer in vector form as above.
It is instructive to write the soln in vector form

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{pmatrix} =
\begin{pmatrix}
  3 - 2\lambda \\
  \lambda \\
  -1 \\
  2
\end{pmatrix}
\]

This shows the solution set is a line in \( \mathbb{R}^4 \) in the direction
Vector form of the solution

It is instructive to write the soln in vector form

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{pmatrix} = \begin{pmatrix}
  3 - 2\lambda \\
  \lambda \\
  -1 \\
  2
\end{pmatrix}
\]

This shows the solution set is a line in $\mathbb{R}^4$ in the direction

**Upshot** This argument shows that you will have as many parameters $\lambda_1, \ldots, \lambda_j$ as you have non-leading columns.
It is instructive to write the soln in vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 - 2\lambda \\ \lambda \\ -1 \\ 2 \end{pmatrix} =$$

This shows the solution set is a line in $\mathbb{R}^4$ in the direction

**Upshot** This argument shows that you will have as many parameters $\lambda_1, \ldots, \lambda_j$ as you have non-leading columns. Geometrically, the solution set is $j$-dimensional.
Vector form of the solution

It is instructive to write the soln in vector form

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{pmatrix} = \begin{pmatrix}
    3 - 2\lambda \\
    \lambda \\
    1 \\
    2
\end{pmatrix}
\]

This shows the solution set is a line in \( \mathbb{R}^4 \) in the direction

**Upshot** This argument shows that you will have as many parameters \( \lambda_1, \ldots, \lambda_j \) as you have non-leading columns. Geometrically, the solution set is \( j \)-dimensional. We usually expect you to write your answer in vector form as above.
There are many sequences of EROs you can apply to obtain row echelon form (and hence solve systems of linear equations). One algorithm is easily seen in the following example. It corresponds to eliminating variables systematically.

First use $R_j = R_j + \alpha_j R_1$, to get zeros in the first column of every row (except row 1):

$$
\begin{bmatrix}
1 & 2 & 3 \\
5 & 0 & -2 \\
-1 & 4 & 11 \\
2 & 10 & 7 \\
6 & & \\
\end{bmatrix}
$$

Now rows 1 and 2 are looking fine. Essentially we eliminated $x_1$ from eqns in $R_2, R_3$.

Next use ERO $R_j = R_j + \beta_j R_2$ to get zeros in the 2nd column of every row (except rows 1 and 2):

$$
R_3 = R_3 - ?
$$

If the matrix were bigger, you would just keep going zeroing one column in turn, until the matrix is in row-echelon form.
Gaussian elimination: easy example

There are many sequences of EROs you can apply to obtain row echelon form (and hence solve systems of linear equations).

\[
\begin{bmatrix}
1 & 2 & 3 & 5 \\
-1 & 4 & 1 & 6 \\
2 & 10 & 7 & 8
\end{bmatrix}
\]

First use \(R_j = R_j + \alpha R_1\), to get zeros in the first column of every row (except row 1):

\[
\begin{bmatrix}
1 & 2 & 3 & 5 \\
0 & 2 & 4 & 11 \\
2 & 10 & 7 & 8
\end{bmatrix}
\]

Now rows 1 and 2 are looking fine. Essentially we eliminated \(x_1\) from eqns in \(R_2, R_3\).

Next use ERO \(R_j = R_j + \beta R_2\) to get zeros in the 2nd column of every row (except rows 1 and 2):

\[
\begin{bmatrix}
1 & 2 & 3 & 5 \\
0 & 2 & 4 & 11 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

If the matrix were bigger, you would just keep going zeroing one column in turn, until the matrix is in row-echelon form.
Gaussian elimination: easy example

There are many sequences of EROs you can apply to obtain row echelon form (and hence solve systems of linear equations). One algorithm is easily seen in the following example. It corresponds to eliminating variables systematically.

\[
\begin{pmatrix}
1 & 2 & 3 \\
5 & -1 & -4 & 1 \\
6 & 2 & 10 & 7
\end{pmatrix}
\]

First use \( R_j = R_j + \alpha_j R_1 \), to get zeros in the first column of every row (except row 1):

\[
\begin{pmatrix}
1 & 2 & 3 \\
5 & 0 & -2 & 4 \\
6 & 2 & 10 & 7
\end{pmatrix}
\]

Now rows 1 and 2 are looking fine. Essentially we eliminated \( x_1 \) from eqns in \( R_2, R_3 \),

Next use ERO \( R_j = R_j + \beta_j R_2 \) to get zeros in the 2nd column of every row (except rows 1 and 2):

\[
\begin{pmatrix}
1 & 2 & 3 \\
5 & 0 & 11 \\
0 & 0 & 0
\end{pmatrix}
\]

If the matrix were bigger, you would just keep going zeroing one column in turn, until the matrix is in row-echelon form.
Gaussian elimination: easy example

There are many sequences of EROs you can apply to obtain row echelon form (and hence solve systems of linear equations). One algorithm is easily seen in the following example. It corresponds to eliminating variables systematically.

First use \( R_j = R_j + \alpha_j R_1 \), to get zeros in the first column of every row (except row 1!):

\[
\begin{pmatrix}
1 & 2 & 3 & 5 \\
-1 & -4 & 1 & 6 \\
2 & 10 & 7 & 8
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 & 5 \\
0 & -2 & 4 & 11 \\
2 & 10 & 7 & 8
\end{pmatrix}
\]

Now rows 1 and 2 are looking fine. Essentially we eliminated \( x_1 \) from eqns in \( R_2, R_3 \).

Next use ERO \( R_j = R_j + \beta_j R_2 \) to get zeros in the 2nd column of every row (except rows 1 and 2):

\[
R_3 = R_3 - ? \rightarrow
\begin{pmatrix}
1 & 2 & 3 & 5 \\
0 & -2 & 4 & 11 \\
2 & 10 & 7 & 8
\end{pmatrix}
\]

If the matrix were bigger, you would just keep going zeroing one column in turn, until the matrix is in row-echelon form.
Gaussian elimination: easy example

There are many sequences of EROs you can apply to obtain row echelon form (and hence solve systems of linear equations). One algorithm is easily seen in the following example. It corresponds to eliminating variables systematically.

First use $R_j = R_j + \alpha_j R_1$, to get zeros in the first column of every row (except row 1!):

$$
\begin{pmatrix}
1 & 2 & 3 & 5 \\
-1 & -4 & 1 & 6 \\
2 & 10 & 7 & 8
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 2 & 3 & 5 \\
0 & -2 & 4 & 11 \\
2 & 10 & 7 & 8
\end{pmatrix}
$$

Now rows 1 and 2 are looking fine.
Gaussian elimination: easy example

There are many sequences of EROs you can apply to obtain row echelon form (and hence solve systems of linear equations). One algorithm is easily seen in the following example. It corresponds to eliminating variables systematically.

First use $R_j = R_j + \alpha_j R_1$, to get zeros in the first column of every row (except row 1!):

\[
\begin{pmatrix}
1 & 2 & 3 & 5 \\
-1 & -4 & 1 & 6 \\
2 & 10 & 7 & 8 \\
\end{pmatrix}
\overset{R_2 = R_2 + ?}{\longrightarrow}
\begin{pmatrix}
1 & 2 & 3 & 5 \\
0 & -2 & 4 & 11 \\
2 & 10 & 7 & 8 \\
\end{pmatrix}
\]

Now rows 1 and 2 are looking fine. Essentially we eliminated $x_1$ from eqns in R2, R3.
Gaussian elimination: easy example

There are many sequences of EROs you can apply to obtain row echelon form (and hence solve systems of linear equations). One algorithm is easily seen in the following example. It corresponds to eliminating variables systematically.

First use $R_j = R_j + \alpha_j R_1$, to get zeros in the first column of every row (except row 1!):

$$
\begin{bmatrix}
1 & 2 & 3 & | & 5 \\
-1 & -4 & 1 & | & 6 \\
2 & 10 & 7 & | & 8
\end{bmatrix}
\xrightarrow{R_2 = R_2 + \cdot}
\begin{bmatrix}
1 & 2 & 3 & | & 5 \\
0 & -2 & 4 & | & 11 \\
2 & 10 & 7 & | & 8
\end{bmatrix}
$$

Now rows 1 and 2 are looking fine. Essentially we eliminated $x_1$ from eqns in R2, R3. Next use ERO $R_j = R_j + \beta_j R_2$ to get zeros in the 2nd column of every row (except rows 1 and 2):

$$
\begin{bmatrix}
1 & 2 & 3 & | & 5 \\
0 & -2 & 4 & | & 11 \\
2 & 10 & 7 & | & 8
\end{bmatrix}
\xrightarrow{R_3 = R_3 - \cdot}
\begin{bmatrix}
1 & 2 & 3 & | & 5 \\
0 & -2 & 4 & | & 11 \\
0 & 0 & 0 & | & 0
\end{bmatrix}
$$
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\begin{pmatrix}
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-1 & -4 & 1 & | & 6 \\
2 & 10 & 7 & | & 8 \\
\end{pmatrix}
$$

$$
R_2 = R_2 + \begin{pmatrix}
0 \\
-2 \\
0 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 2 & 3 & | & 5 \\
0 & -2 & 4 & | & 11 \\
0 & 0 & 7 & | & 8 \\
\end{pmatrix}
$$

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$$
R_3 = R_3 - \begin{pmatrix}
0 \\
0 \\
-2 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 2 & 3 & | & 5 \\
0 & -2 & 4 & | & 11 \\
0 & 0 & 0 & | & 0 \\
\end{pmatrix}
$$

If the matrix were bigger, you would just keep going zeroing one column in turn, until the matrix is in row-echelon form.
The above algorithm is fine until you meet a matrix like
\[
\begin{bmatrix}
0 & 2 & 3 \\
5 & 2 & -2 \\
1 & 3 & 2
\end{bmatrix}
\]
or
\[
\begin{bmatrix}
0 & 0 & 3 \\
5 & 0 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]
Here you need to swap the first row with the second or third.

In general

1. Go to the 1st (from left) nonzero column called the pivot column.
2. Go down that column to the 1st nonzero entry called the pivot element.
3. Swap that row, called the pivot row, with the first (unless of course it is the first).
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\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
0 & 0 & 3 & 5 \\
0 & 0 & 4 & 1 \\
0 & 1 & 2 & 6
\end{pmatrix}
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\end{pmatrix}
\] or \[
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0 & 0 & 3 & 5 \\
0 & 0 & 4 & 1 \\
0 & 1 & 2 & 6
\end{pmatrix}
\]

Here you need to swap the first row with the second or third.

In general

**Pivot elements**

1. Go to the 1st (from left) nonzero column called the *pivot column*. 
The above algorithm is fine until you meet a matrix like

\[
\begin{pmatrix}
0 & 2 & 3 & 5 \\
2 & -2 & 4 & 1 \\
1 & 3 & 2 & 6
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
0 & 0 & 3 & 5 \\
0 & 0 & 4 & 1 \\
0 & 1 & 2 & 6
\end{pmatrix}
\]

Here you need to swap the first row with the second or third.

In general, to perform gaussian elimination:

**Pivot elements**

1. Go to the 1st (from left) nonzero column called the *pivot column*.
2. Go down that column to the 1st nonzero entry called the *pivot element*.
The above algorithm is fine until you meet a matrix like

\[
\begin{pmatrix}
0 & 2 & 3 & 5 \\
2 & -2 & 4 & 1 \\
1 & 3 & 2 & 6
\end{pmatrix} \quad \text{or} \quad 
\begin{pmatrix}
0 & 0 & 3 & 5 \\
0 & 0 & 4 & 1 \\
0 & 1 & 2 & 6
\end{pmatrix}
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In general

Pivot elements

1. Go to the 1st (from left) nonzero column called the *pivot column*.
2. Go down that column to the 1st nonzero entry called the *pivot element*.
3. Swap that row, called the *pivot row* with the first (unless of course it is the first).
Gaussian elimination: complete algorithm

Algorithm

To solve \((A|b)\):

1. Select the pivot element.
2. Swap the pivot row to the top if necessary.
3. Reduce to zero all the entries below the pivot element using EROs.
4. Repeat steps 1, 2, and 3 on the submatrix of rows and columns to the right and below the pivot element...recursively until you run out of pivot elements!
5. When you arrive at row-echelon form, use back-substitution to solve.

Example.

Solve

\[
\begin{pmatrix}
1 & -1 & 1 \\
-1 & 1 & -1 \\
-2 & -1 & 1 \\
1 & 0 & 2 \\
1 & 3 & -1 \\
0 & -1 & 5 \\
\end{pmatrix}
\]
Gaussian elimination: complete algorithm

Algorithm
To solve $(A | b)$:

1. Select the pivot element.
2. Swap the pivot row to the top if necessary.
3. Reduce to zero all the entries below the pivot element using EROs.
4. Repeat steps 1, 2 and 3 on the submatrix of rows and columns to the right and below the pivot element...recursively until you run out of pivot elements!
5. When you arrive at row-echelon form, use back-substitution to solve.

Example.
Solve
\[
\begin{pmatrix}
1 & -1 & 1 & -2 \\
1 & 1 & 2 & 2 \\
1 & 0 & 2 & 1 \\
1 & 3 & -1 & 5 
\end{pmatrix}
\]
## Gaussian elimination: complete algorithm

### Algorithm

To solve \((A|\mathbf{b})\):

1. Select the pivot element.

### Example

Solve

\[
\begin{bmatrix}
1 & -1 & 1 \\
-1 & 1 & -2 \\
-1 & 1 & 1 \\
-2 & 1 & 0 \\
1 & 0 & 2 \\
1 & 3 & -1 \\
0 & 1 & 5
\end{bmatrix}
\]
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Example. Solve

$$
\begin{pmatrix}
1 & -1 & 1 \\
-1 & 1 & -2 \\
-2 & -1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
3 \\
-1 \\
\end{pmatrix}
$$
Gaussian elimination: complete algorithm

Algorithm

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5. When you arrive at row-echelon form, use back-substitution to solve.

Example. Solve

\[
\begin{pmatrix}
1 & -1 & 1 & -1 & -2 \\
-1 & 1 & -1 & 1 & 2 \\
1 & 0 & 2 & 1 & 1 \\
3 & -1 & 5 & 1 & 0 \\
\end{pmatrix}
\]
Do the lines $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \lambda \in \mathbb{R}$, $x = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \mu \in \mathbb{R}$ intersect?
Q Do the lines

\[ \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R} \]

\[ \mathbf{x} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad \mu \in \mathbb{R} \]

intersect?
Q Do the lines

\[
x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R} \quad x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad \mu \in \mathbb{R}
\]

intersect?

You can turn this into a vector equation in \( \lambda \) and \( \mu \):

\[
\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}
\]

\[
\lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix}.
\]
As we saw before, this corresponds to a system of linear eqns which, in augmented matrix form is

\[
\begin{pmatrix}
-1 & -2 & -1 \\
1 & 1 & -3 \\
-1 & -2 & -2
\end{pmatrix}
\]

\[R_2 = R_2 + R_1\]
\[R_3 = R_3 - R1\]

\[
\begin{pmatrix}
-1 & -2 & -1 \\
0 & -1 & -4 \\
0 & 0 & -1
\end{pmatrix}
\]

For this, and many other problems, what you want to know is 'Does a solution exist?', not 'What is the solution?' You can read this straight off the row-echelon form!

The system is inconsistent because the bottom row gives the inconsistent eqn

\[0 \lambda + 0 \mu = -1\]

No intersection so the lines are skew.

Remark: An inconsistent eqn corresponds to a row in the augmented matrix of the form

\[(0 \ 0 \ \ldots \ 0 \ | \ b)\]

where \(b \neq 0\).

For \((U \ | \ y)\) in row echelon form, this occurs precisely when \(y\) is a leading column as above.
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\end{pmatrix}
\Rightarrow
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1 & 1 & -3 \\
-1 & -2 & -2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
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0 & 0 & -1
\end{pmatrix}
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\end{pmatrix}
\rightarrow \begin{align*}
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For \((U|y)\) in row echelon form, this occurs precisely when \(y\) is a leading column as above.
Nature of solutions and row echelon form

Theorem
Suppose that the system \((A|b)\) has equivalent row-echelon form \((U|y)\).

The system has no solution iff \(y\) is a leading column iff it has an inconsistent row.

If \(y\) is not a leading column then
1. The system has a unique solution if \(U\) has no non-leading columns, and
2. The system has infinitely many solutions if \(U\) has non-leading columns.

There is one arbitrary parameter for each non-leading column.

E.g.
Describe geometrically, the solution sets to
\[
\begin{bmatrix}
-1 & 3 & 1 \\
6 & 0 & 2 & 3 \\
0 & 0 & 7 & 8
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
-1 & 3 & 1 \\
6 & 0 & 2 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Nature of solutions and row echelon form

Back-substitution readily gives

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0 & 2 & 3 \\
0 & 0 & 7
\end{pmatrix}
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Nature of solutions and row echelon form

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Nature of solutions and row echelon form

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  1. The system has a unique solution if \(U\) has no non-leading columns, and

Back-substitution readily gives

**Theorem**

Suppose that the system $(A|\mathbf{b})$ has equivalent row-echelon form $(U|\mathbf{y})$.

- The system has no solution iff $\mathbf{y}$ is a leading column iff it has an inconsistent row.
- If $\mathbf{y}$ is not a leading column then
  1. The system has a unique solution if $U$ has no non-leading columns, and
  2. The system has infinitely many solutions if $U$ has non-leading columns.
Back-substitution readily gives

**Theorem**

Suppose that the system \((A|b)\) has equivalent row-echelon form \((U|y)\).

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**E.g.** Describe geometrically, the solution sets to

$$
\begin{pmatrix}
-1 & 3 & 1 & 6 \\
0 & 2 & 3 & 5 \\
0 & 0 & 7 & 8 \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
-1 & 3 & 1 & 6 \\
0 & 2 & 3 & 5 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
$$
Underdetermined equations

Question.
Can the system
\[
\begin{bmatrix}
1 & -1 & 1 & -2 & 2 & -1 & 3 & 8 \\
-2 & -3 & 3 & 5
\end{bmatrix}
\]
have a unique solution?

Solution.
In general, if you have more variables than equations you cannot have a unique soln.
Indeed, in the row-echelon form (U | y), no. leading columns = no. leading rows < no. columns in U so there must be a non-leading column in U.
If y is leading then
If y is non-leading then

Daniel Chan (UNSW)
Chapter 4: Linear Equations
**Question.** Can the system

\[
\begin{pmatrix}
1 & -1 & 1 & -1 & -2 \\
2 & -2 & -1 & 3 & 7 \\
-2 & -3 & 3 & 8 & 5
\end{pmatrix}
\]

have a unique solution?
Underdetermined equations

Question. Can the system

$$\begin{pmatrix}
1 & -1 & 1 & -1 & -2 \\
2 & -2 & -1 & 3 & 7 \\
-2 & -3 & 3 & 8 & 5 \\
\end{pmatrix}$$

have a unique solution?

Solution.
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\]

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\end{pmatrix}
\]

have a unique solution?

Solution. In general, if you have more variables than equations you cannot have a unique solution. Indeed, in the row-echelon form \((U|y)\),

\[
\text{no. leading columns} = \text{no. leading rows} < \text{no. columns in } U
\]

so there must be a non-leading column in \(U\).
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\[
\begin{pmatrix}
1 & -1 & 1 & -1 & -2 \\
2 & -2 & -1 & 3 & 7 \\
-2 & -3 & 3 & 8 & 5
\end{pmatrix}
\]

have a unique solution?

**Solution.** In general, if you have more variables than equations you cannot have a unique soln. Indeed, in the row-echelon form \((U|y)\),

\[
\text{no. leading columns} = \text{no. leading rows} < \text{no. columns in } U
\]

so there must be a non-leading column in \(U\).

If \(y\) is leading then
Underdetermined equations

**Question.** Can the system

\[
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1 & -1 & 1 & -1 & -2 \\
2 & -2 & -1 & 3 & 7 \\
-2 & -3 & 3 & 8 & 5 \\
\end{pmatrix}
\]

have a unique solution?

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so there must be a non-leading column in \(U\).

If \(y\) is leading then

If \(y\) is non-leading then
Describing spans in cartesian form

Which \( \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3 \) are in the span of

\( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \), \( \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \) and \( \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \)?

As a vector eqn, this is asking whether there exist \( \lambda_1, \lambda_2, \lambda_3 \) such that

\[
\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}
\]

In augmented matrix form this is

\[
\begin{pmatrix}
1 & 1 & 3 & b_1 \\
2 & -1 & 3 & b_2 \\
-2 & -3 & 4 & b_3
\end{pmatrix}
\]
Describing spans in cartesian form

Q Which \( \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3 \) are in the span of \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \) and \( \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \)?
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\[
\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + ...
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In augmented matrix form this is

\[
\begin{pmatrix} 1 & 1 & 3 & b_1 \\ 2 & -1 & 3 & b_2 \\ 3 & -2 & 4 & b_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 3 & b_1 \\ 0 & -3 & -3 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + \frac{b_1}{3} - \frac{5b_2}{3} \end{pmatrix}
\]
We need to know whether this system has a solution. The right-hand side vector here is a leading column — unless $b_3 + b_1^3 - 5b_2^3 = 0$.

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$ is a linear combination of the three vectors precisely when $(\ast)$ holds.

The span of the 3 vectors is the plane in $b_1b_2b_3$-space defined by the cartesian eqn $(\ast)$.

Can you see what is happening geometrically?
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Can you see what is happening geometrically?
We often solve equations like $\sin x = 1/2$ or more generally $f(x) = c$ for some $f: \mathbb{R} \rightarrow \mathbb{R}$ and constant $c \in \mathbb{R}$.

Solving simultaneous equations can be put in this framework if we allow vector-valued functions $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Consider $a_1, \ldots, a_n \in \mathbb{R}$ and the function $l: \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $l(x) = a_1 x_1 + \cdots + a_n x_n$.

This is a scalar valued linear function (with vector inputs).

$l(x) = b$ is a linear equation.

Given a vector $x \in \mathbb{R}^n$ and an $m \times n$-matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ we define $A x = \begin{pmatrix} a_{11} x_1 + \cdots + a_{1n} x_n \\ \vdots \\ a_{m1} x_1 + \cdots + a_{mn} x_n \end{pmatrix}$.

$l: \mathbb{R}^n \rightarrow \mathbb{R}^m$: $x \mapsto A x$ i.e. defined by $l(x) = A x$ is an example of a vector-valued linear function.

The system of linear equations corresponding to $(A|b)$ is equivalent to the vector equation $A x = b$.

We call $A x$ the matrix-vector product of $A$ and $x$. 
Solving simultaneous equations: functional viewpoint

We often solve eqns like \( \sin x = \frac{1}{2} \) or more generally \( f(x) = c \) for some \( f : \mathbb{R} \rightarrow \mathbb{R} \) and constant \( c \in \mathbb{R} \).
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Consider $a_1, \ldots, a_n \in \mathbb{R}$ and the function $l : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $l(x) = a_1 x_1 + \ldots + a_n x_n$. This is a scalar valued linear function (with vector inputs). $l(x) = b$ is a linear eqn. Given a vector $x \in \mathbb{R}^n$ and an $m \times n$-matrix $A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$ we define $A x = \begin{pmatrix} a_{11} x_1 + \ldots + a_{1n} x_n \\ \vdots \\ a_{m1} x_1 + \ldots + a_{mn} x_n \end{pmatrix}$. $l : \mathbb{R}^n \rightarrow \mathbb{R}^m : x \mapsto A x$ i.e. defined by $l(x) = A x$ is an example of a vector-valued linear function. The system of linear eqns corresponding to $(A | b)$ is equivalent to the vector equation $A x = b$. We call $A x$ the matrix-vector product of $A$ and $x$. 

Daniel Chan (UNSW)

Chapter 4: Linear Equations

Semester 1 2018
We often solve eqns like \( \sin x = \frac{1}{2} \) or more generally \( f(x) = c \) for some \( f : \mathbb{R} \to \mathbb{R} \) and constant \( c \in \mathbb{R} \). Solving simultaneous eqns can be put in this framework if we allow vector-valued functions \( f : \mathbb{R}^n \to \mathbb{R}^m \).

Consider \( a_1, \ldots, a_n \in \mathbb{R} \) and the function \( l : \mathbb{R}^n \to \mathbb{R} \) defined by

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l(x) = a_1x_1 + \ldots + a_nx_n.
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This is a *scalar valued linear function* (with vector inputs).
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\[
A = \begin{pmatrix}
a_{11} & \ldots & a_{1n} \\
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\end{pmatrix}
\]

we define \( Ax = \begin{pmatrix} a_{11} x_1 + \ldots + a_{1n} x_n \\
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\( l : \mathbb{R}^n \rightarrow \mathbb{R}^m : x \mapsto Ax \) i.e. defined by \( l(x) = Ax \) is an example of a vector-valued linear function.

The system of linear eqns corresponding to \((A|b)\) is equivalent to the vector equation \( Ax = b \). We call \( Ax \) the matrix-vector product of \( A \) and \( x \).
Matrix-vector product: examples

Calculate \[
\begin{pmatrix}
1 & 3 \\
1 & 2 \\
4 & 0
\end{pmatrix}
\begin{pmatrix}
2 \\
-1
\end{pmatrix}.
\]

Let \( A = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \).

Describe the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : x \mapsto Ax \) as a mapping from the plane to itself.

Hence solve \( f(x) = x \). We’ll see much more of this next semester.
Q Calculate

\[
\begin{pmatrix}
1 & 3 \\
1 & 2 \\
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\[
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1 & 3 \\
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\end{pmatrix}
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2 \\
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\]

Q Let \( A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Describe the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : \mathbf{x} \mapsto A\mathbf{x} \) as a mapping from the plane to itself. Hence solve \( f(\mathbf{x}) = \mathbf{x} \). We’ll see much more of this next semester.
Distributive law for matrix-vector product

Part of the reason we use the product notation/terminology is the Distributive Law.

Suppose that $x, y \in \mathbb{R}^n$ and $A$ is an $m \times n$-matrix. Then

$$A(x + y) = Ax + Ay.$$ 

Why?

For ease of notation, suppose $n = 2$ so $x = (x_1, x_2)$, $y = (y_1, y_2)$.

$$A(x + y) = A(x_1 + y_1, x_2 + y_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{m1} & a_{m2} \end{pmatrix} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{m1}x_1 + a_{m2}x_2 \end{pmatrix} + \begin{pmatrix} a_{11}y_1 + a_{12}y_2 \\ a_{m1}y_1 + a_{m2}y_2 \end{pmatrix} = Ax + Ay.$$ 

Ex

Prove that if $\lambda \in \mathbb{R}$, $A(\lambda x) = \lambda (Ax)$. 

Daniel Chan (UNSW)

Chapter 4: Linear Equations

Semester 1 2018 33 / 42
Distributive law for matrix-vector product

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\[
A(x + y) = A(x_1 + y_1, x_2 + y_2) = \begin{pmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22} \\
    \vdots & \vdots \\
    a_{m1} & a_{m2}
\end{pmatrix}
\begin{pmatrix}
    x_1 + y_1 \\
    x_2 + y_2
\end{pmatrix} =
\begin{pmatrix}
    a_{11}x_1 + a_{12}y_1 \\
    a_{21}x_1 + a_{22}y_1 \\
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\end{pmatrix} +
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\]

Exercise

Prove that if \( \lambda \in \mathbb{R} \), then \( A(\lambda x) = \lambda (Ax) \).
Part of the reason we use the product notation/terminology is the Distributive Law.

Suppose that \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \) and \( A \) is an \( m \times n \)-matrix. Then

\[
A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}.
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**Distributive Law**

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Part of the reason we use the product notation/terminology is the **Distributive Law**

Suppose that \( x, y \in \mathbb{R}^n \) and \( A \) is an \( m \times n \)-matrix. Then

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A(x + y) = Ax + Ay.
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**Why?** For ease of notation, suppose \( n = 2 \) so \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \).
Distributive law for matrix-vector product

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**Distributive Law**

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\[
A(\mathbf{x} + \mathbf{y}) = A\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = \begin{pmatrix} a_{11}(x_1 + y_1) + a_{12}(x_2 + y_2) \\ \\ \\ a_m(x_1 + y_1) + a_m(x_2 + y_2) \\ a_{11}x_1 + a_{12}x_2 \\ \\ \\ a_mx_1 + a_mx_2 \end{pmatrix} + \begin{pmatrix} a_{11}y_1 + a_{12}y_2 \\ \\ \\ a_my_1 + a_my_2 \\ \end{pmatrix} = A\mathbf{x} + A\mathbf{y}
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  a_{m1}(x_1 + y_1) + a_{m2}(x_2 + y_2)
\end{pmatrix} = \begin{pmatrix}
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  \vdots \\
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  a_{m1}y_1 + a_{m2}y_2
\end{pmatrix} = Ax + Ay
\]

**Ex** Prove that if \( \lambda \in \mathbb{R} \), \( A(\lambda x) = \lambda (Ax) \).
Next semester, we will study the concept of linearity in great generality. For now, we say a function \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is linear if for all \( x, y \in \mathbb{R}^n \), \( \lambda \in \mathbb{R} \) we have

\[
T(x + y) = T(x) + T(y),
\]

\[
T(\lambda x) = \lambda T(x).
\]

Last slide \( \Rightarrow \) the function \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m : x \mapsto A x \) is linear. Next semester, we'll see all linear maps \( \mathbb{R}^n \rightarrow \mathbb{R}^m \) have this form. E.g.

\[
T(x, y) = x^2 + y
\]

Warning: Our definition conflicts with the one in calculus! \( f(x) = ax + b \) is linear iff

Sometimes, we call such functions affine linear.
Next semester, we will study the concept of linearity in great generality.

For now, we say a function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if for all $x, y \in \mathbb{R}^n, \lambda \in \mathbb{R}$ we have $T(x + y) = T(x) + T(y)$, $T(\lambda x) = \lambda T(x)$.

Last slide $\Rightarrow$ the function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m: x \mapsto Ax$ is linear.

Next semester, we'll see all linear maps $\mathbb{R}^n \rightarrow \mathbb{R}^m$ have this form.

E.g. $T(x, y) = x^2 + y$

Warning: Our definition conflicts with the one in calculus!

$f(x) = ax + b$ is linear iff

Sometimes, we call such functions affine linear.
Linearity

Next semester, we will study the concept of linearity in great generality. For now, we say a function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if for all $x, y \in \mathbb{R}^n, \lambda \in \mathbb{R}$ we have

$T(x + y) = T(x) + T(y)$,

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Homogeneous equations: abstract view

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear function and $b \in \mathbb{R}^m$. We consider the linear equation $T(x) = b$ and solve for $x$. We say $T(x) = b$ is homogeneous if $b = 0$, and is inhomogeneous otherwise.

**Proposition**
Suppose $x_1, \ldots, x_r$ are solutions to the homogeneous eqn $T(x) = 0$. Then

1. $x = 0$ is a homogeneous solution (existence).
2. $x_1 + x_2$ and $\lambda x_1$ are homogeneous solutions for any $\lambda \in \mathbb{R}$.
3. Any linear combination $\lambda_1 x_1 + \cdots + \lambda_r x_r$ is a homogeneous solution.

**Proof.**
For (1) just calculate. For (2) and (3), use induction.

**Terminology**
We express (2) by saying that the set of homogeneous solutions is closed under addition and scalar multiplication. We express (3) by saying homogeneous solutions are closed under linear combinations.
Homogeneous equations: abstract view

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**Proof.**

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For 2) just calculate:

\[
T(x_1 + x_2) = T x_1 + T x_2 = 0 + 0 = 0.
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Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear function and \( b \in \mathbb{R}^m \). We consider the linear equation \( T\mathbf{x} = b \) and solve for \( \mathbf{x} \).

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**Proposition**

Suppose \( \mathbf{x}_1, \ldots, \mathbf{x}_r \) are solutions to the homogeneous eqn \( T\mathbf{x} = 0 \).

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We express (2) by saying that the set of homogeneous solns is closed under addition and scalar multiplication. We express (3) by saying homog solns are closed under linear combinations.
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Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear function and $b \in \mathbb{R}^m$. We consider the linear equation $Tx = b$ and solve for $x$.

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Suppose $x_1, \ldots, x_r$ are solutions to the homogeneous eqn $Tx = 0$.

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The solutions to an inhomogeneous linear equation \( T \mathbf{x} = \mathbf{b} \) are related to the solutions of the corresponding homogeneous equation \( T \mathbf{x} = 0 \), as long as solutions exist!

**Proposition**

Let \( \mathbf{x} = \mathbf{x}_p \) be a solution to \( T \mathbf{x} = \mathbf{b} \).

Then \( \mathbf{x} = \mathbf{x}_p + \mathbf{x}_h \) is also a solution for any homogeneous solution \( \mathbf{x}_h \) to \( T \mathbf{x} = 0 \).

Every solution to \( T \mathbf{x} = \mathbf{b} \) has this form.

**Remark**

The importance of the proposition is that if you know a particular inhomogeneous solution, and the general homogeneous solution, then you know the general inhomogeneous solution too.

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It suffices to show \( \mathbf{x}_h \) is a homogeneous solution:

**Corollary**

If \( \mathbf{x} = \mathbf{0} \) is the unique homogeneous solution, then an inhomogeneous solution is unique (assuming it exists).
Inhomogeneous equations: abstract view

The solutions to an inhomogeneous linear eqn $Tx = b$ are related to the solns of the corresponding homogeneous eqn $Tx = 0$.

Proposition

Let $x = x_p$ be a soln to $Tx = b$. Then $x = x_p + x_h$ is also a solution for any homog soln $x_h$ to $Tx = 0$.

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Inhomogeneous equations: abstract view

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**Corollary** If $\mathbf{x} = \mathbf{0}$ is the unique homogeneous soln, then an inhomogeneous soln is unique (assuming it exists).
Relating the abstract view with result from Gaussian elimination

Consider the linear eqn

$$A\mathbf{x} = \mathbf{b}$$

corresponding to the reduced row echelon

$$
\begin{bmatrix}
1 & 2 & 0 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

Assigning parameters $x_2 = \lambda, x_4 = \mu$, back-substitution gives the general soln as

$$
\mathbf{x} = \begin{bmatrix}
8 \\
0 \\
3 \\
0
\end{bmatrix}
+ \lambda \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix}
+ \mu \begin{bmatrix}
-4 \\
0 \\
-2 \\
1
\end{bmatrix}
$$

Setting $\lambda = \mu = 0$ gives the particular soln

$$
\begin{bmatrix}
8 \\
0 \\
3 \\
0
\end{bmatrix}
$$

and the propn last slide $\Rightarrow$ the general homog soln is the span of
Relating the abstract view with result from Gaussian elimination

**E.g.** Consider the linear eqn $Ax = b$ corresponding to the reduced row echelon augmented matrix

\[
\begin{bmatrix}
1 & 2 & 0 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Assigning parameters $x_2 = \lambda$, $x_4 = \mu$, back-substitution gives the general soln as

\[
\begin{bmatrix}
8 \\
0 \\
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0 & 0 & 0 & 0 & 0
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$$

Assigning parameters $x_2 = \lambda$, $x_4 = \mu$, back-substitution gives the general soln as

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\begin{pmatrix}
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3 \\
0
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Daniel Chan (UNSW)
Chapter 4: Linear Equations
Semester 1 2018 37 / 42
Relating the abstract view with result from Gaussian elimination

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Are there any polynomials \( p(x) = a_0 + a_1 x + \ldots + a_n x^n \) which satisfy the differential eqn
\[ p''(x) + 3xp'(x) - p(x) = x + 2. \]

A The two sides can only be equal if all the coefficients match. Now
\[ xp'(x) = a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots + na_n x^n \]
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Application: differential equations

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Coeff of 1: \[ 2a_2 + 3 \times 0 - a_0 = 2 \]
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In augmented matrix form, for the variables \(a_0, \ldots, a_n\):

\[
\begin{pmatrix}
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0 & 2 & 0 & 6 & 0 & \vdots & \vdots \\
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Note that uniqueness of the soln is independent of the entries in the last column, which matches up with our result relating homogeneous and inhomogeneous solns.
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Q Is the line with parametric equation \( x = \begin{pmatrix} -9 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \) parallel to the plane through the points with coords \((0, 0, 0), (1, 0, 1), (2, -1, 2)\)?
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Modern economic problem

The Republic produces 3 types of X-wing fighters: 1) 1 person, 2) 2 person and 3) 4 person. Production involves 3 plants operated by Gungans, Ewoks and Wookiees, which deal with a) weapons manufacture, b) engine manufacture and c) assembly respectively.

The time (in hours) required to produce each type of fighter is given below:

<table>
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The Gungans will only work 20 hours a week, the Ewoks 33 and the Wookiees 44. If the production plants are to operate at full capacity, how many of each type of fighter will be produced?

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Let \( x_1 \), \( x_2 \), \( x_3 \) be the number of 1 person, 2 person and 4 person fighters produced a week.

We have one equation for each plant.

Chapter 4: Linear Equations

Semester 1 2018
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