Chapter 4: Linear Equations

Daniel Chan (based on Ian Doust’s notes)

UNSW

Semester 1 2015
An Ancient Chinese Problem

The following maths problem was taken from the Chinese maths text "Nine Chapters on the Mathematical Arts" ca. 200CE

Problem

One pint of good wine costs 50 gold pieces, one pint of bad only 10. If two pints are bought for 30 gold pieces, how much of each wine is bought?

The Chinese knew this is a simple algebra problem. If \( x \), \( y \) are the amount (in pints) of good and bad wine bought then

\[
\begin{align*}
\text{eq1:} & \quad x + y = 2 \\
\text{eq2:} & \quad 50x + 10y = 30
\end{align*}
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We review the solution, as we wish to generalise this later.

Key

Eliminate variables.

This can be done in lots of ways but the method we'll generalise is

Subtract $50 \times \text{eqn (1)}$ from eqn (2) to get

$$-40y = -70 \Rightarrow y = 7/4.$$

Now obtain $x$ by using this and eqn (1) (or (2))

$$x = 2 - y = 1\frac{1}{4}.$$
Kindergarten solution

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Linear equations

A linear equation in \( n \) unknowns \( x_1, \ldots, x_n \) is an equation of the form

\[
a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b
\]

where \( a_1, \ldots, a_n, b \) are given real numbers.

E.g.

The name linear comes from

We are interested in solving several of these simultaneously as in the ancient Chinese problem.

A system of \( m \) linear equations in \( n \) variables is

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\begin{align*}
  a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
  a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\
  \vdots \quad \vdots \quad \vdots & \quad \vdots \\
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A linear equation in $n$ unknowns $x_1, \ldots, x_n$ is an equation of the form

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Point normal form for lines in $\mathbb{R}^2$

Just as there are point normal forms for planes in $\mathbb{R}^3$, there are point normal forms for lines in $\mathbb{R}^2$ (can you see the general pattern??)

Let $L \subset \mathbb{R}^2$ be the line in $\mathbb{R}^2$ passing through the point $a$ with normal vector $n = (n_1, n_2)$.

$L$ is the set of points $x$ satisfying the point normal form $n \cdot (x - a) = 0 \iff n_1 x_1 + n_2 x_2 = n \cdot x = n \cdot a$.

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Intersecting 2 lines in $\mathbb{R}^2$

Consider solving a system of 2 eqns in 2 unknowns $x, y$:

$$a_{11}x + a_{12}y = b_1 (3)$$
$$a_{21}x + a_{22}y = b_2 (4)$$

Plotting solutions in the $xy$-plane allows us to understand the nature of solutions better.

Assume both eqns are non-zero (otherwise ??) so eqns (3), (4) define lines $L_1, L_2$ and $L_1 \cap L_2$ represents the simultaneous soln.

On geometric grounds, we see there are ?? possibilities.

Let's verify this algebraically!
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2 linear equations in 2 unknowns: algebraic version

$L_1 = L_2$ iff eqn (3) and (4) are scalar multiples of each other. Simultaneous soln same as the soln to one of them.

$L_1 \parallel L_2$ but not equal. This means that the normals are parallel so

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = r \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix},$ for some $r \in \mathbb{R}$.

Then $(3) - r \times (4)$ gives

$0 = b_1 - rb_2 \neq 0$ WHY?

so there are no solns. ($\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$ are not parallel.

Our procedure gives a unique soln. This is easy to see from any e.g. such as in the ancient Chinese problem.
$L_1 = L_2$ iff

$\text{this means that the normals are parallel so}\ (a_{11}a_{12}) = r(a_{21}a_{22})$, for some $r \in \mathbb{R}$.

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Intersecting 3 planes in $\mathbb{R}^3$

Here are some possibilities for intersecting 3 planes $P_1, P_2, P_3 \subset \mathbb{R}^3$.

- The planes intersect in a point.
- There is no intersection as the three are parallel and distinct.
- $P_1, P_2$ intersect in a line which is parallel to $P_3$. There's no intersection.
- $P_1 = P_2$ intersect $P_3$ in a line so the three intersect in a line.

etc

There are heaps more cases, which makes it hard to analyse. We need make systematic, our method of solving simultaneous solns and understanding the set of solns.

This procedure, described in the “Nine Chapters” and re-discovered by Gauss is called Gaussian elimination.

Aside Gauss developed it to solve a system of 6 linear eqns in 6 unknowns and hence compute the orbit of the dwarf planet Ceres.

Look up Stigler’s law on Wikipedia.
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Augmented matrix form

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$$
\begin{align*}
    a_{11}x_1 & + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
    a_{21}x_1 & + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
    \vdots & \quad \vdots \quad \vdots \\
    a_{m1}x_1 & + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{align*}
$$

Here the notation omits the unknowns $x_1, \ldots, x_n$ which must be understood from the context (or irrelevant).
Augmented matrix form

We first abbreviate our system of $m$ linear equations in $n$ variables:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \vdots & \quad \vdots & \quad \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

To augmented matrix form:

\[
\begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
    a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{pmatrix}
\]
Augmented matrix form

We first abbreviate our system of $m$ linear equations in $n$ variables:

$$
\begin{align*}
\begin{cases}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \vdots & \vdots & \vdots & \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{cases}
\end{align*}
$$

to augmented matrix form:

$$
\begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
    a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{pmatrix}
$$

Here the notation omits the unknowns $x_1, \ldots, x_n$ which must be understood from the context (or irrelevant).
Example: augmented matrix form

Write the augmented matrix form for the system

\[
\begin{align*}
6x_1 - 2x_2 + 5x_3 &= 7 \\
3x_1 + x_2 + 2x_3 &= 0
\end{align*}
\]

Solution.

E.g.
Write the system of equations with augmented matrix form
\[
\left[
\begin{array}{ccc|c}
2 & 1 & 1 & 7 \\
-3 & 3 & 5 & 0
\end{array}
\right]
\]

Daniel Chan (based on Ian Doust’s notes) (UNSW)

Chapter 4: Linear Equations
Semester 1 2015
Example: augmented matrix form

E.g. Write the augmented matrix form for the system

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\end{align*}
\]

**Solution.**

**E.g** Write the system of equations with augmented matrix form

\[
\begin{pmatrix}
2 & 1 & 1 & -3 \\
3 & 5 & 0 & 7
\end{pmatrix}.
\]
Matrices

A matrix is just a rectangular array of numbers, like 
\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 7 & 5
\end{pmatrix}
\]

We would say that this is a $2 \times 3$ matrix as it has 2 rows and 3 columns.

An $m \times 1$ matrix just looks like a vector 
\[
b = \begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{pmatrix} \in \mathbb{R}^m
\]

The augmented matrix consists of the coefficient matrix (on the left of the line) augmented by the right-hand-side vector. We can write 
\[
\begin{pmatrix}
2 & 1 & 1 \\
-3 & 3 & 0 \\
7 & &
\end{pmatrix}
\]
as 
\[
\left( A \mid b \right) \quad \text{where} \quad A = \begin{pmatrix}
2 & 1 & 1 \\
3 & 5 & 0
\end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix}
-3 \\
7
\end{pmatrix}
\]
Matrices

A *matrix* is just a rectangular array of numbers, like

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 7 & 5
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\]

We would say that this is a $2 \times 3$ matrix as it has 2 rows and 3 columns.

An $m \times 1$ matrix just looks like a vector $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$.

The augmented matrix consists of the coefficient matrix (on the left of the line) augmented by the right-hand-side vector. We can write

\[
\begin{pmatrix}
2 & 1 & 1 \\
-3 & 3 & 0
\end{pmatrix}
\]

as

\[
\begin{pmatrix}
A \\
b
\end{pmatrix}
\]

where $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 5 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$.
A matrix is just a rectangular array of numbers, like

\[
\begin{pmatrix}
1 & 2 & 3 \\
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\[
\begin{pmatrix}
2 & 1 & 1 \\
-3 & 3 & 0 \\
7 & & \\
\end{pmatrix}
\]
as

\[
\begin{pmatrix}
A \\
b
\end{pmatrix}
\]
where $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 5 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$.
A *matrix* is just a rectangular array of numbers, like

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An $m \times 1$ matrix just looks like a vector $b = 
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{pmatrix} \in \mathbb{R}^m$

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An $m \times 1$ matrix just looks like a vector $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$.

The augmented matrix consists of the *coefficient matrix* (on the left of the line) augmented by the right-hand-side vector. We can write \[
\begin{pmatrix}
2 & 1 & 1 \\
3 & 3 & 0
\end{pmatrix}
\begin{pmatrix}
-3 \\
7
\end{pmatrix}
\] as $(A|b)$ where $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 3 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$.
Simultaneous equations as a single vector equation

Any system of (not necessarily linear) simultaneous eqns can be written as a single vector eqn. The above eqn \( \begin{pmatrix} A \\ b \end{pmatrix} \) can be re-written as

\[
\begin{pmatrix}
-3 \\
7 \\
\end{pmatrix}
= \begin{pmatrix}
2x_1 + x_2 + x_3 \\
3x_1 + 5x_2 \\
\end{pmatrix}
= x_1 \begin{pmatrix}
2 \\
3 \\
\end{pmatrix}
+ x_2 \begin{pmatrix}
1 \\
5 \\
\end{pmatrix}
+ x_3 \begin{pmatrix}
1 \\
0 \\
\end{pmatrix}
\]

or perhaps as

\[
x_1 a_1 + x_2 a_2 + x_3 a_3 = b.
\]

where \( a_1, a_2, a_3 \) are the columns of \( A \).

That is, solving this system is equivalent to asking:

How can we write \( b \) as a linear combination of the vectors \( a_1, a_2, a_3 \)?
Any system of (not necessarily linear) simultaneous eqns can be written as a single vector eqn.
Simultaneous equations as a single vector equation

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The above eqn \((A|\mathbf{b})\) can be re-written as

\[
\begin{pmatrix} -3 & 7 \\
2x_1 + x_2 + x_3 & 3x_1 + 5x_2 \\
\end{pmatrix} = \begin{pmatrix} x_1 \\
2x_3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\
1 & 5 \\
1 & 0 \end{pmatrix}.
\]

or perhaps as

\[
x_1 a_1 + x_2 a_2 + x_3 a_3 = \mathbf{b},
\]

where \(a_1, a_2, a_3\) are the columns of \(A\).

That is, solving this system is equivalent to asking:

How can we write \(\mathbf{b}\) as a linear combination of the vectors \(a_1, a_2, \text{ and } a_3\)?
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Any system of (not necessarily linear) simultaneous eqns can be written as a single vector eqn.

The above eqn \((A|b)\) can be re-written as

\[
\begin{pmatrix} -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 + x_3 \\ 3x_1 + 5x_2 \end{pmatrix} = x_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
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7
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3x_1 + 5x_2
\end{pmatrix} = x_1 \begin{pmatrix}
2 \\
3
\end{pmatrix} + x_2 \begin{pmatrix}
1 \\
5
\end{pmatrix} + x_3 \begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

or perhaps as

\[x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3 = \mathbf{b}.\]
Simultaneous equations as a single vector equation

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or perhaps as

\[x_1 a_1 + x_2 a_2 + x_3 a_3 = b.\]

where \(a_1, a_2, a_3\) are the columns of \(A\).
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x_1 a_1 + x_2 a_2 + x_3 a_3 = b.
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That is, solving this system is equivalent to asking:

*How can we write \(b\) as a linear combination of the vectors \(a_1, a_2 and a_3\)?*
Equivalent systems of linear equations

We say a system of linear eqns is **consistent** if it has at least one solution and **inconsistent** otherwise.

E.g. \(x + y = 2\), \(2x + 2y = 5\) is consistent.

**Definition**

Two systems of linear equations \((A|b), (A'|b')\) are said to be **equivalent** if they have exactly the same set of solutions.

The following define equivalent systems of linear equations

\[
\begin{align*}
x + y &= 2 \\
2x + y &= 2 \\ (5)
\end{align*}
\]

\[
\begin{align*}
2x + 3y &= 5 \\
3x + 4y &= 7 \\ (6)
\end{align*}
\]

Why? Note that the 2nd system is gotten from the 1st by adding eqn (5)LHS to eqn (6)LHS. Similarly, the 1st system...
Equivalent systems of linear equations

We say a system of linear eqns is *consistent* if it has at least one solution and *inconsistent* otherwise.
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\] (5)

\[
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\end{align*}
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The systems are equivalent because
Elementary row operations

To solve \((A|b)\), we pass to an equivalent system of eqns where one of the eqns has only 1 unknown. We can solve this and then by induction on the number of variables, should be done.

**Q** How do you pass to an equivalent system of linear equations?

Any of the following procedures will yield an equivalent system:

- **Elementary row operations**
- Swap the order in which you write the equations!
- Multiply any equation by a **nonzero** constant. (Why non-zero?)
- Add a multiple of one equation to another equation. For example, you can replace equation (5) with equation (5) + 7 × equation (6).

These are called the **elementary row operations**.

This is all easier to do if we use the augmented matrix form! We just replace the word eqn above with row of the augmented matrix.
Elementary row operations

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  - equation (5)
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equation (5)
with

  \[
  \text{equation (5) + 7 \times equation (6)}.
  \]

These are called the *elementary row operations*.

This is all easier to do if we use the augmented matrix form! We just replace the word eqn above with row of the augmented matrix.
Notation for EROs

We record the elementary row operation on an arrow between the old system and the new. In each case below, the system on the right is equivalent to the left.

\[
\begin{bmatrix}
0 & 2 & 3 \\
-1 & 3 & 1 \\
2 & 4 & 7 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-1 & 3 & 1 \\
0 & 2 & 3 \\
2 & 4 & 7 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 3 \\
-1 & -4 & 1 \\
2 & 10 & 7 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 \\
0 & -2 & 4 \\
2 & 10 & 7 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 0 & 3 & 6 \\
0 & 0 & 0 & 4 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 0 & 3 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
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We record the elementary row operation on an arrow between the old system and the new. In each case below, the system on the right is equivalent to the left.

\[
\begin{pmatrix}
0 & 2 & 3 & | & 5 \\
-1 & 3 & 1 & | & 6 \\
2 & 4 & 7 & | & 8 \\
\end{pmatrix} \quad \rightarrow \quad R_2 \leftrightarrow R_1 \quad \begin{pmatrix}
-1 & 3 & 1 & | & 6 \\
0 & 2 & 3 & | & 5 \\
2 & 4 & 7 & | & 8 \\
\end{pmatrix}.
\]

\[
\begin{pmatrix}
1 & 2 & 3 & | & 5 \\
-1 & -4 & 1 & | & 6 \\
2 & 10 & 7 & | & 8 \\
\end{pmatrix} \quad \rightarrow \quad R_2 = R_2 + R_1 \quad \begin{pmatrix}
1 & 2 & 3 & | & 5 \\
0 & -2 & 4 & | & 11 \\
2 & 10 & 7 & | & 8 \\
\end{pmatrix}.
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & | & 8 \\
0 & 0 & 3 & 6 & | & 9 \\
0 & 0 & 0 & 4 & | & 8 \\
\end{pmatrix} \quad \rightarrow \quad R_2 = \frac{1}{3} R_2 \quad \begin{pmatrix}
1 & 2 & 3 & 4 & | & 8 \\
0 & 0 & 1 & 2 & | & 3 \\
0 & 0 & 0 & 1 & | & 2 \\
\end{pmatrix}.
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & | & 8 \\
0 & 0 & 3 & 6 & | & 9 \\
0 & 0 & 0 & 4 & | & 8 \\
\end{pmatrix} \quad \rightarrow \quad R_3 = \frac{1}{4} R_3 \quad \begin{pmatrix}
1 & 2 & 3 & 4 & | & 8 \\
0 & 0 & 1 & 2 & | & 3 \\
0 & 0 & 0 & 1 & | & 2 \\
\end{pmatrix}.
\]
We wish to apply a series of EROs until we arrive at a system of linear eqns we can solve!

Which systems are these?

Ones like the bottom right one on the previous slide.

Remember, that augmented matrix is just saying

\[ \begin{align*}
  x_1 &+ 2x_2 &+ 3x_3 &+ 4x_4 &= 8 \\
  x_3 &+ 2x_4 &= 3 \\
  x_4 &= 2
\end{align*} \]

Starting from the bottom, you can first solve \( x_4 \), and then moving up solve \( x_3 \) and then (see later) \( x_2 \), \( x_1 \).

Such a system is said to be in row-echelon form.

Solving it by starting from the bottom eqn and progressively moving up is called back substitution.
We wish to apply a series of EROs until we arrive at a system of linear eqns we can solve!
We wish to apply a series of EROs until we arrive at a system of linear eqns we can solve!

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Row echelon form: motivation

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\[
\begin{align*}
 x_1 & \quad + \quad 2x_2 & \quad + \quad 3x_3 & \quad + \quad 4x_4 & \quad = \quad 8 \\
 x_3 & \quad + \quad 2x_4 & \quad & \quad = \quad 3 \\
 x_4 & \quad & \quad & \quad = \quad 2
\end{align*}
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Starting from the bottom, you can first solve \(x_4\),
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Starting from the bottom, you can first solve \( x_4 \), and then moving up solve \( x_3 \) and then (see later) \( x_2, x_1 \).
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Such a system is said to be in \textit{row-echelon form}. 
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Starting from the bottom, you can first solve \(x_4\), and then moving up solve \(x_3\) and then (see later) \(x_2, x_1\).

Such a system is said to be in row-echelon form. Solving it by starting from the bottom eqn and progressively moving up is called back substitution.
Row echelon form

In any matrix

1. A leading row is one which is not all zeros,
2. the leading entry in a leading row is the first (i.e. leftmost) non-zero entry,
3. a leading column is a column which contains the leading entry for some row.

Row echelon form

A matrix is said to be in row-echelon form if
1. all leading rows are above all non-leading rows (so any all-zero rows are at the bottom of the matrix), and
2. in every leading row, the leading entry is further to the right than the leading entry in any row higher up in the matrix.

If you remove the zeros, row echelon form gives an upside-down staircase shape.

A matrix is in reduced row echelon form if it
1) is in row echelon form,
2) all leading entries are 1 and,
3) all entries above leading entries are 0.

Daniel Chan (based on Ian Doust’s notes) (UNSW)
Chapter 4: Linear Equations
Semester 1 2015
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A matrix is in *reduced row echelon form* if it 1) is in row echelon form, 2) all leading entries are 1 and, 3) all entries above leading entries are 0.
Examples: leading columns, row echelon form

E.g. Check the following is in row echelon form and solve:

\[
\begin{pmatrix}
1 & 2 \\
3 & 0 & 2
\end{pmatrix}
\]

E.g. \[
\begin{pmatrix}
1 & 2 & 3 \\
-1 & 0 & 0 & 2
\end{pmatrix}
\]
Examples: leading columns, row echelon form

E.g. Check the following is in row echelon form and solve

$$
\begin{pmatrix}
1 & 2 & 3 \\
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\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 2 & 4 \\
\end{pmatrix}
\]

E.g.

\[
\begin{pmatrix}
1 & 2 & 3 & -1 \\
0 & 0 & 2 & \pi \\
0 & 0 & e & 0 \\
\end{pmatrix}
\]
Back-substitution

We can solve systems of linear eqns if they are in row echelon form as in the following e.g.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\((R_1)\) \((R_2)\) \((R_3)\)

From \((R_3)\), \(x_4 = 2\).

Back substituting this into \((R_2)\) you get \(x_3 = \_\).

The convention is that you set the variable for any non-leading column to be an arbitrary parameter.

So, let \(x_2 = \lambda\).

Then \(x_1 = 8 - 2\lambda - 3 \times (-1) - 4 \times 2 = 3 - 2\lambda\).
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We can solve systems of linear eqns if they are in row echelon form as in the following e.g.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 8 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 2 \\
\end{pmatrix}
\]

(R1)  
(R2)  
(R3)

From (R3), \(x_4 = 2\).

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$$\begin{pmatrix} 1 & 2 & 3 & 4 & 8 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad (R1)$$

$$\begin{pmatrix} \end{pmatrix} \quad (R2)$$

$$\begin{pmatrix} \end{pmatrix} \quad (R3)$$

From $(R3)$, $x_4 = 2$. Back substituting this into $(R2)$ you get $x_3 = \lambda$.

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\[
x_1 = 8 - 2\lambda - 3 \times (-1) - 4 \times 2 = 3 - 2\lambda.
\]
Vector form of the solution

It is instructive to write the solution in vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ \lambda \\ \lambda \end{pmatrix}$$

This shows the solution set is a line in $\mathbb{R}^4$ in the direction of $\lambda$. This argument shows that you will have as many parameters $\lambda_1, \ldots, \lambda_j$ as you have non-leading columns. Geometrically, the solution set is $j$-dimensional.

We usually expect you to write your answer in vector form as above.
It is instructive to write the soln in vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ \lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 2 \end{pmatrix}$$

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  2
\end{pmatrix} =
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-1 \\
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  \lambda \\
  -1 \\
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\end{pmatrix} =
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**Upshot** This argument shows that you will have as many parameters \( \lambda_1, \ldots, \lambda_j \) as you have non-leading columns.
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  \lambda \\
  -1 \\
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    x_4
\end{pmatrix}
=
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    \lambda \\
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    2
\end{pmatrix}
\]

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**Upshot** This argument shows that you will have as many parameters \(\lambda_1, \ldots, \lambda_j\) as you have non-leading columns. Geometrically, the solution set is \(j\)-dimensional. We usually expect you to write your answer in vector form as above.
Gaussian elimination: easy example

There are many sequences of EROs you can apply to obtain row echelon form (and hence solve systems of linear equations). One algorithm is easily seen in the following example. It corresponds to eliminating variables systematically.

First use $R_j = R_j + \alpha_j R_1$, to get zeros in the first column of every row (except row 1):

\[
\begin{pmatrix}
1 & 2 & 3 \\
5 & -1 & -4 & 1
\end{pmatrix}
\]

$R_2 \rightarrow R_2 - R_1$

Now rows 1 and 2 are looking fine. Essentially we eliminated $x_1$ from eqns in $R_2$, $R_3$.

Next use ERO $R_j = R_j + \beta_j R_2$ to get zeros in the 2nd column of every row (except rows 1 and 2):

$R_3 \rightarrow R_3 - R_2$

If the matrix were bigger, you would just keep going zeroing one column in turn, until the matrix is in row-echelon form.
Gaussian elimination: easy example

There are many sequences of EROs you can apply to obtain row echelon form (and hence solve systems of linear equations).

\[
\begin{bmatrix}
1 & 2 & 3 \\
5 & 0 & -2 \\
-4 & 1 & 11
\end{bmatrix}
\]

\[
R_2 = R_2 - R_1 \rightarrow
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & -2 \\
-2 & 4 & 11
\end{bmatrix}
\]

\[
R_3 = R_3 + 2R_2 \rightarrow
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & -2 \\
0 & 0 & 0
\end{bmatrix}
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First use $R_j = R_j + \alpha R_1$, to get zeros in the first column of every row (except row 1!):

$$
\begin{pmatrix}
1 & 2 & 3 \\
5 & -1 & -4 \\
6 & 2 & 10 \\
8 & & \\
\end{pmatrix}
$$

$R_2 \rightarrow R_2 - \rightarrow$

$$
\begin{pmatrix}
1 & 2 & 3 \\
0 & -2 & 4 \\
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8 & & \\
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R_3 \rightarrow R_3 - \rightarrow
$$

$$
\begin{pmatrix}
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0 & 0 & 0 \\
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\[
\begin{pmatrix}
1 & 2 & 3 & 5 \\
-1 & -4 & 1 & 6 \\
2 & 10 & 7 & 8
\end{pmatrix} \xrightarrow{R_2 = R_2} \begin{pmatrix}
1 & 2 & 3 & 5 \\
0 & -2 & 4 & 11
\end{pmatrix}
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1 & 2 & 3 & 5 \\
0 & -2 & 4 & 11 \\
2 & 10 & 7 & 8 \\
\end{pmatrix}
\]

Now rows 1 and 2 are looking fine. Essentially we eliminated \( x_1 \) from eqns in \( R_2, R_3 \). Next use ERO \( R_j = R_j + \beta_j R_2 \) to get zeros in the 2nd column of every row (except rows 1 and 2):

\[
R_3 = R_3 \xrightarrow{} \begin{pmatrix}
1 & 2 & 3 & 5 \\
0 & -2 & 4 & 11 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
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\[
\begin{pmatrix}
1 & 2 & 3 & | & 5 \\
-1 & -4 & 1 & | & 6 \\
2 & 10 & 7 & | & 8 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 & | & 5 \\
0 & -2 & 4 & | & 11 \\
0 & 0 & 0 & | & 0 \\
\end{pmatrix}
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\[
R_3 = R_3
\rightarrow
\begin{pmatrix}
1 & 2 & 3 & | & 5 \\
0 & -2 & 4 & | & 11 \\
0 & 0 & 0 & | & 0 \\
\end{pmatrix}
\]

If the matrix were bigger, you would just keep going zeroing one column in turn, until the matrix is in row-echelon form.
The above algorithm is fine until you meet a matrix like

\[
\begin{bmatrix}
0 & 2 & 3 \\
5 & 2 & -2 \\
1 & 3 & 2 \\
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
0 & 0 & 3 \\
5 & 0 & 0 \\
1 & 0 & 1 \\
6 & 0 & 2 \\
\end{bmatrix}
\]

Here you need to swap the first row with the second or third.

In general, you can follow these steps:

1. Go to the 1st (from left) nonzero column called the **pivot column**.
2. Go down that column to the 1st nonzero entry called the **pivot element**.
3. Swap that row, called the **pivot row**, with the first (unless of course it is the first).
The above algorithm is fine until you meet a matrix like

$$
\begin{pmatrix}
0 & 2 & 3 \\
5 & -2 & 4 \\
1 & 3 & 2 \\
\end{pmatrix}
$$

or

$$
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0 & 1 & 2 \\
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Pivots

The above algorithm is fine until you meet a matrix like

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0 & 2 & 3 & 5 \\
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\end{pmatrix}
\]

or

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\begin{pmatrix}
0 & 0 & 3 & 5 \\
0 & 0 & 4 & 1 \\
0 & 1 & 2 & 6 \\
\end{pmatrix}
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\end{pmatrix}
\text{ or }
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0 & 0 & 4 & 1 \\
0 & 1 & 2 & 6
\end{pmatrix}
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Daniel Chan (based on Ian Doust’s notes) (UNSW)
Pivots

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1 & 3 & 2 & 6
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\end{pmatrix}
\]

Here you need to swap the first row with the second or third.

In general

Pivot elements

1. Go to the 1st (from left) nonzero column called the *pivot column*.
The above algorithm is fine until you meet a matrix like
\[
\begin{pmatrix}
0 & 2 & 3 & 5 \\
2 & -2 & 4 & 1 \\
1 & 3 & 2 & 6
\end{pmatrix}
\text{ or }
\begin{pmatrix}
0 & 0 & 3 & 5 \\
0 & 0 & 4 & 1 \\
0 & 1 & 2 & 6
\end{pmatrix}
\]
Here you need to swap the first row with the second or third.
In general

**Pivot elements**

1. Go to the 1st (from left) nonzero column called the *pivot column*.
2. Go down that column to the 1st nonzero entry called the *pivot element*.
Pivots

The above algorithm is fine until you meet a matrix like

$$\begin{pmatrix} 0 & 2 & 3 & 5 \\ 2 & -2 & 4 & 1 \\ 1 & 3 & 2 & 6 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 3 & 5 \\ 0 & 0 & 4 & 1 \\ 0 & 1 & 2 & 6 \end{pmatrix}$$

Here you need to swap the first row with the second or third.

In general

**Pivot elements**

1. Go to the 1st (from left) nonzero column called the *pivot column*.
2. Go down that column to the 1st nonzero entry called the *pivot element*.
3. Swap that row, called the *pivot row* with the first (unless of course it is the first).
Gaussian elimination: complete algorithm

Algorithm

To solve \((A|b)\):

1. Select the pivot element.
2. Swap the pivot row to the top if necessary.
3. Reduce to zero all the entries below the pivot element using EROs.
4. Repeat steps 1, 2 and 3 on the submatrix of rows and columns to the right and below the pivot element...recursively until you run out of pivot elements!
5. When you arrive at row-echelon form, use back-substitution to solve.

Example.
Solve
\[
\begin{bmatrix}
1 & -1 & 1 & -1 & 2 \\
2 & -2 & -1 & 3 & 4 \\
1 & 0 & 3 & 6 & 34 \\
3 & -3 & 3 & 8 & 39 \\
\end{bmatrix}
\]
Algorithm

To solve $(A|b)$:

1. Select the pivot element.
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Example. Solve

$$
\begin{pmatrix}
1 & -1 & 1 & -1 \\
2 & -2 & -1 & 3 \\
1 & 0 & 3 & 6 \\
7 & -3 & 3 & 8
\end{pmatrix}
$$

Daniel Chan (based on Ian Doust’s notes) (UNSW)
Algorithm

To solve \((A|\mathbf{b})\):

1. Select the pivot element.
Gaussian elimination: complete algorithm

Algorithm

To solve \((A|b)\):

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\begin{pmatrix}
1 & -1 & 1 & -1 & -2 \\
2 & -2 & -1 & 3 & 7 \\
1 & 0 & 3 & 6 & 34 \\
4 & -3 & 3 & 8 & 39
\end{pmatrix}
\]
Application: geometry

Do the lines 

\[ x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda \in \mathbb{R} \]

\[ x = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad \mu \in \mathbb{R} \]

intersect? You can turn this into a vector equation in \( \lambda \) and \( \mu \):

\[ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \]

\[ \lambda \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}. \]
Q Do the lines

\[ \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R} \quad \quad \quad \mathbf{x} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad \mu \in \mathbb{R} \]

intersect?
Q Do the lines

\[ \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R} \]

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\[ \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix}. \]
As we saw before, this corresponds to a system of linear eqns which, in augmented matrix form is
\[
\begin{pmatrix}
-1 & 0 & -1 & 1 & 1 \\
-2 & -1 & 0 & -1 & -2 \\
\end{pmatrix}
\]
\[R_2 = R_2 + R_1\]
\[R_3 = R_3 - R_1\]

For this, and many other problems, what you want to know is 'Does a solution exist?', not 'What is the solution?' You can read this straight off the row-echelon form!

The system is inconsistent because the bottom row gives the inconsistent eqn
\[0 \lambda + 0 \mu = -1.\]

No intersection so the lines are skew.

Remark: An inconsistent eqn corresponds to a row in the augmented matrix of the form
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & | & b \\
\end{pmatrix}
\]
where \(b \neq 0\).

For \((U | y)\) in row echelon form, this occurs precisely when \(y\) is a leading column as above.
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-1 & -2 & -1 \\
1 & 1 & -3 \\
-1 & -2 & -2
\end{pmatrix}
\]
\[
R_2 = R_2 + R1
\]
\[
R_3 = R_3 - R1
\]
\[
\begin{pmatrix}
-1 & -2 & -1 \\
0 & -1 & -4 \\
0 & 0 & -1
\end{pmatrix}
\]

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\[
(0 \ 0 \ ... \ 0 | b)
\]
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\[
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-1 & -2 & -2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
R_2 = R_2 + R1 \\
R_3 = R_3 - R1
\end{pmatrix}
\begin{pmatrix}
-1 & -2 & -1 \\
0 & -1 & -4 \\
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\end{pmatrix}
\]

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The system is inconsistent because the bottom row gives the inconsistent eqn
\[0\lambda + 0\mu = -1.\]

No intersection so the lines are skew.
Application cont’d

As we saw before, this corresponds to a system of linear eqns which, in augmented matrix form is

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\begin{pmatrix}
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1 & 1 & -3 \\
-1 & -2 & -2
\end{pmatrix}
\begin{pmatrix}
R_2 = R_2 + R1 \\
R_3 = R_3 - R1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-1 & -2 & -1 \\
0 & -1 & -4 \\
0 & 0 & -1
\end{pmatrix}
\]

For this, and many other problems, what you want to know is ‘Does a solution exist?’, not ‘What is the solution?’

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No intersection so the lines are skew.

**Remark:** An inconsistent eqn corresponds to a row in the augmented matrix of the form

\[(0 \ 0 \ldots 0 \mid b)\] where \(b \neq 0.\)
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For \((U | y)\) in row echelon form, this occurs precisely when \(y\) is a leading column as above.
Nature of solutions and row echelon form

Suppose that the system \((A|b)\) has equivalent row-echelon form \((U|y)\).

- The system has no solution iff \(y\) is a leading column iff it has an inconsistent row.
- If \(y\) is not a leading column then
  - The system has a unique solution if \(U\) has no non-leading columns, and
  - The system has infinitely many solutions if \(U\) has non-leading columns. There is one arbitrary parameter for each non-leading column.

E.g. Describe geometrically, the solution sets to
\[
\begin{pmatrix}
-1 & 3 & 1 \\
6 & 0 & 2 & 3 \\
0 & 0 & 7 & 8
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
-1 & 3 & 1 \\
6 & 0 & 2 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
Back-substitution readily gives

\[
\begin{bmatrix}
-1 & 3 & 1 & 6 \\
0 & 2 & 3 & 5 \\
0 & 0 & 0 & 0 \\
0 & 0 & 7 & 8
\end{bmatrix}
\]
Back-substitution readily gives

**Theorem**

Suppose that the system $(A|b)$ has equivalent row-echelon form $(U|y)$.

- The system has no solution iff $y$ is a leading column iff it has an inconsistent row.
- If $y$ is not a leading column then
  1. The system has a unique solution if $U$ has no non-leading columns, and
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Nature of solutions and row echelon form

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**Theorem**

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-1 & 3 & 1 & \vphantom{0} & 6 \\
0 & 2 & 3 & \vphantom{0} & 5 \\
0 & 0 & 7 & \vphantom{0} & 8
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
-1 & 3 & 1 & \vphantom{0} & 6 \\
0 & 2 & 3 & \vphantom{0} & 5 \\
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\end{pmatrix}
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**Theorem**

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Back-substitution readily gives

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Suppose that the system \((A|b)\) has equivalent row-echelon form \((U|y)\).

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\[
\begin{pmatrix}
-1 & 3 & 1 & 6 \\
0 & 2 & 3 & 5 \\
0 & 0 & 7 & 8 \\
\end{pmatrix}
\text{ and } \begin{pmatrix}
-1 & 3 & 1 & 6 \\
0 & 2 & 3 & 5 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Underdetermined equations

Question.
Can the system
\[
\begin{bmatrix}
1 & -1 & 1 & -2 & 2 & -1 & 3 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
have a unique solution?

Solution.
In general, if you have more variables than equations you cannot have a unique soln.
Indeed, in the row-echelon form \((U | y)\),
no. leading columns = no. leading rows < no. columns in \(U\).
If \(y\) is leading then
If \(y\) is non-leading then

Daniel Chan (based on Ian Doust’s notes) (UNSW)

 Chapter 4: Linear Equations
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**Question.** Can the system

\[
\begin{pmatrix}
1 & -1 & 1 & -1 & -2 \\
2 & -2 & -1 & 3 & 7 \\
-2 & -3 & 3 & 8 & 5
\end{pmatrix}
\]

have a unique solution?
Question. Can the system

\[
\begin{pmatrix}
1 & -1 & 1 & -1 & -2 \\
2 & -2 & -1 & 3 & 7 \\
-2 & -3 & 3 & 8 & 5
\end{pmatrix}
\]

have a unique solution?

Solution.
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Solution. In general, if you have more variables than equations you cannot have a unique soln. Indeed, in the row-echelon form \((U|y)\),

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so there must be a non-leading column in \(U\).
Question. Can the system

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\begin{pmatrix}
1 & -1 & 1 & -1 & -2 \\
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\end{pmatrix}
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have a unique solution?

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\[
\text{no. leading columns} = \text{no. leading rows} < \text{no. columns in } U
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so there must be a non-leading column in \(U\).

If \(y\) is leading then
**Question.** Can the system

\[
\begin{pmatrix}
1 & -1 & 1 & -1 & -2 \\
2 & -2 & -1 & 3 & 7 \\
-2 & -3 & 3 & 8 & 5
\end{pmatrix}
\]

have a unique solution?

**Solution.** In general, if you have more variables than equations you cannot have a unique soln. Indeed, in the row-echelon form \((U|y)\),

\[
\text{no. leading columns = no. leading rows < no. columns in } U
\]

so there must be a non-leading column in \(U\).

If \(y\) is leading then

If \(y\) is non-leading then
Describing spans in cartesian form

Which \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3 \text{ are in the span of } \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}?

As a vector eqn, this is asking whether there exist \( \lambda_1, \lambda_2, \lambda_3 \) such that

\[ \lambda_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + \lambda_3 \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \]

In augmented matrix form this is

\[
\begin{bmatrix}
1 & 1 & 3 & b_1 \\
2 & -1 & 3 & b_2 \\
-2 & -2 & 4 & b_3
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 1 & 3 & b_1 \\
0 & 0 & -3 & -3b_2 \\
0 & 0 & 0 & b_3 - 3b_1 + 5b_2
\end{bmatrix}
\]
Describing spans in cartesian form

Q Which \((b_1, b_2, b_3) \in \mathbb{R}^3\) are in the span of \((1, 2, 3), (1, -1, -2)\) and \((3, 3, 4)\)?
Describing spans in cartesian form

Q Which \( (b_1, b_2, b_3) \in \mathbb{R}^3 \) are in the span of \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \) and \( \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \)?

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Describing spans in cartesian form

Q Which \( \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3 \) are in the span of \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \) and \( \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \)?

A As a vector eqn, this is asking whether there exist \( \lambda_1, \lambda_2, \lambda_3 \) such that

\[ \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \]
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A As a vector eqn, this is asking whether there exist \( \lambda_1, \lambda_2, \lambda_3 \) such that

\[
\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \lambda_3 \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}
\]

In augmented matrix form this is

\[
\begin{pmatrix}
1 & 1 & 3 & | & b_1 \\
2 & -1 & 3 & | & b_2 \\
3 & -2 & 4 & | & b_3 \\
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 1 & 3 & | & b_1 \\
0 & -3 & -3 & | & b_2 - 2b_1 \\
0 & 0 & 0 & | & b_3 + \frac{b_1}{3} - \frac{5b_2}{3} \\
\end{pmatrix}
\]
We need to know whether this system has a solution.

The right-hand side vector here is a leading column — unless

\[ b_3 + b_{13} - 5b_2 = 0 \]  

(\star)

\[
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3 
\end{pmatrix}
\]
is a linear combination of the three vectors precisely when (\star) holds.

The span of the 3 vectors is the plane in \( b_1 b_2 b_3 \) -space defined by the cartesian eqn (\star).

Can you see what is happening geometrically?
We need to know whether this system has a solution.

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\begin{bmatrix}
 b_1 \\
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 b_3 \\
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Can you see what is happening geometrically?
Solving simultaneous equations: functional viewpoint

We often solve eqns like \( \sin x = \frac{1}{2} \) or more generally \( f(x) = c \) for some \( f : \mathbb{R} \to \mathbb{R} \) and constant \( c \in \mathbb{R} \).

Solving simultaneous eqns can be put in this framework if we allow vector-valued functions \( f : \mathbb{R}^n \to \mathbb{R}^m \).

Consider \( a_1, \ldots, a_n \in \mathbb{R} \) and the function \( l : \mathbb{R}^n \to \mathbb{R} \) defined by \( l(x) = a_1 x_1 + \ldots + a_n x_n \).

This is a scalar valued linear function (with vector inputs).

\( l(x) = b \) is a linear eqn.

Given a vector \( x \in \mathbb{R}^n \) and an \( m \times n \)-matrix \( A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \), we define \( A x = \begin{pmatrix} a_{11} x_1 + \ldots + a_{1n} x_n \\ \vdots \\ a_{m1} x_1 + \ldots + a_{mn} x_n \end{pmatrix} \).

\( l : \mathbb{R}^n \to \mathbb{R}^m : x \mapsto Ax \) i.e. defined by \( l(x) = Ax \) is an example of a vector-valued linear function.

The system of linear eqns corresponding to \((A|b)\) is equivalent to the vector equation \( Ax = b \).

We call \( Ax \) the matrix-vector product of \( A \) and \( x \).
Solving simultaneous equations: functional viewpoint

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Daniel Chan (based on Ian Doust’s notes) (UNSW)
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The system of linear eqns corresponding to \( (A|b) \) is equivalent to the vector equation \( Ax = b \). We call \( Ax \) the matrix-vector product of \( A \) and \( x \).
Matrix-vector product: examples

Q Calculate
\[
\begin{pmatrix}
1 & 3 \\
1 & 2 \\
4 & 0
\end{pmatrix}
\]
\[
(2 - 1)
\]

Let
\[A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\]

Describe the function
\[f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : x \mapsto Ax\]
as a mapping from the plane to itself.

Hence solve \[f(x) = x\]. We'll see much more of this next semester.

Daniel Chan (based on Ian Doust’s notes) (UNSW)

Chapter 4: Linear Equations

Semester 1 2015
Q Calculate

\[
\begin{pmatrix}
1 & 3 \\
1 & 2 \\
4 & 0
\end{pmatrix}
\begin{pmatrix}
2 \\
-1
\end{pmatrix}.
\]
Matrix-vector product: examples

Q Calculate

$$\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$ 

Q Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. 

Matrix-vector product: examples

Q Calculate

\[
\begin{pmatrix}
1 & 3 \\
1 & 2 \\
4 & 0
\end{pmatrix}
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-1
\end{pmatrix}.
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Distributive law for matrix-vector product

Part of the reason we use the product notation/terminology is the Distributive Law

Suppose that \( x, y \in \mathbb{R}^n \) and \( A \) is an \( m \times n \)-matrix. Then

\[
A(x + y) = Ax + Ay.
\]

Why?

For ease of notation, suppose \( n = 2 \) (else you just need more ...)

\[
A(x + y) = A(x_1 + y_1, x_2 + y_2) = \begin{pmatrix}
a_{11}(x_1 + y_1) + a_{12}(x_2 + y_2) \\
a_{21}(x_1 + y_1) + a_{22}(x_2 + y_2) \\
\vdots \\
a_{m1}(x_1 + y_1) + a_{m2}(x_2 + y_2)
\end{pmatrix}
= \begin{pmatrix}
a_{11}x_1 + a_{12}x_2 \\
a_{21}x_1 + a_{22}x_2 \\
\vdots \\
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\end{pmatrix} + \begin{pmatrix}
a_{11}y_1 + a_{12}y_2 \\
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Part of the reason we use the product notation/terminology is the **Distributive Law**

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Distributive Law

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Why? For ease of notation, suppose $n = 2$ (else you just need more ...)

$$A(x + y) = A \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = \begin{pmatrix} a_{11}(x_1 + y_1) + a_{12}(x_2 + y_2) \\ \vdots \\ a_{m1}(x_1 + y_1) + a_{m2}(x_2 + y_2) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 \end{pmatrix} + \begin{pmatrix} a_{11}y_1 + a_{12}y_2 \\ \vdots \\ a_{m1}y_1 + a_{m2}y_2 \end{pmatrix} = Ax + Ay$$
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\]

**Ex** Prove that if \( \lambda \in \mathbb{R} \), \( A(\lambda x) = \lambda(Ax) \).
Linearity

Next semester, we will study the concept of linearity in great generality.

For now, we say a function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if for all $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ we have
\[
T(x + y) = T(x) + T(y),
\]
\[
T(\lambda x) = \lambda T(x).
\]

Last slide $\Rightarrow$ the function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$: $x \mapsto \rightarrow Ax$ is linear.

Next semester, we'll see all linear maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$ have this form.

E.g. $T(x, y) = x^2 + y$.

Warning: Our definition conflicts with the one in calculus! $f(x) = ax + b$ is linear iff sometimes, we call such functions affine linear.
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\]

Last slide \( \Longrightarrow \) the function \( T : \mathbb{R}^n \to \mathbb{R}^m : x \mapsto Ax \) is linear. Next semester, we’ll see all linear maps \( \mathbb{R}^n \to \mathbb{R}^n \) have this form.

\textbf{E.g.} \( T \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y \)

\textbf{Warning} Our defn conflicts with the one in calculus! \( f(x) = ax + b \) is linear iff
Next semester, we will study the concept of linearity in great generality. For now, we say a function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *linear* if for all $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ we have

$$T(x + y) = T(x) + T(y), \quad T(\lambda x) = \lambda T(x).$$

Last slide $\implies$ the function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m : x \mapsto Ax$ is linear. Next semester, we’ll see all linear maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$ have this form.

**E.g.** $T \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y$

**Warning** Our defn conflicts with the one in calculus! $f(x) = ax + b$ is linear iff

Sometimes, we call such functions *affine linear*. 
Homogeneous equations: abstract view

Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear function and \( b \in \mathbb{R}^m \).

We consider the linear equation \( Tx = b \) and solve for \( x \).

We say \( T x = b \) is homogeneous if \( b = 0 \), and is inhomogeneous otherwise.

**Proposition**

Suppose \( x_1, \ldots, x_r \) are solutions to the homogeneous eqn \( T x = 0 \).

1. \( x = 0 \) is a homogeneous soln, (existence).
2. \( x_1 + x_2, \lambda x_1 \) are homog solns for any \( \lambda \in \mathbb{R} \).
3. Any linear combination \( \lambda_1 x_1 + \cdots + \lambda_r x_r \) is a homog soln.

**Proof.**

For (1) just calculate.  
For 2) just calculate:  
\[
T(x_1 + x_2) = T x_1 + T x_2 = 0 + 0 = 0.
\]
Similarly,  
\[
T(\lambda x_1) = \lambda (T x_1) = \lambda 0 = 0.
\]

Terminology

We express (2) by saying that the set of homogeneous solns is closed under addition and scalar multiplication. We express (3) by saying homog solns are closed under linear combinations.
Homogeneous equations: abstract view

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear function and $b \in \mathbb{R}^m$. 

1. $x = 0$ is a homogeneous solution (existence).
2. $x_1 + x_2$ and $\lambda x_1$ are homogeneous solutions for any $\lambda \in \mathbb{R}$.
3. Any linear combination $\lambda_1 x_1 + \ldots + \lambda_r x_r$ is a homogeneous solution.

Proof. For (1) just calculate. For (2) just calculate:

$T(x_1 + x_2) = T x_1 + T x_2 = 0 + 0 = 0.$

Similarly, $T(\lambda x_1) = \lambda T x_1 = \lambda 0 = 0$.

Terminology
We express (2) by saying that the set of homogeneous solutions is closed under addition and scalar multiplication. We express (3) by saying that homogeneous solutions are closed under linear combinations.
Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear function and $b \in \mathbb{R}^m$. We consider the *linear equation* $Tx = b$ and solve for $x$.
Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear function and $b \in \mathbb{R}^m$. We consider the linear equation $Tx = b$ and solve for $x$.
We say $Tx = b$ is homogeneous if $b = 0$,

**Proposition**

Suppose $x_1, \ldots, x_r$ are solutions to the homogeneous eqn $Tx = 0$. Then:

1. $x = 0$ is a homogeneous soln, (existence).
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**Proof.**

For (1) just calculate. For (2) just calculate:

$$T(x_1 + x_2) = T(x_1) + T(x_2) = 0 + 0 = 0.$$ 
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**Terminology**

We express (2) by saying that the set of homogeneous solns is closed under addition and scalar multiplication. We express (3) by saying that homog solns are closed under linear combinations.
Homogeneous equations: abstract view

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear function and $b \in \mathbb{R}^m$. We consider the linear equation $T \mathbf{x} = b$ and solve for $\mathbf{x}$.

We say $T \mathbf{x} = b$ is homogeneous if $b = 0$, and is inhomogeneous otherwise.

**Proposition**

Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_r$ are solutions to the homogeneous eqn $T \mathbf{x} = 0$.
Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear function and \( b \in \mathbb{R}^m \). We consider the linear equation \( Tx = b \) and solve for \( x \).

We say \( Tx = b \) is *homogeneous* if \( b = 0 \), and is *inhomogeneous* otherwise.

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Proposition

Suppose $x_1, \ldots, x_r$ are solutions to the homogeneous eqn $Tx = 0$.

1. $x = 0$ is a homogeneous soln, (existence).
2. $x_1 + x_2, \lambda x_1$ are homog solns for any $\lambda \in \mathbb{R}$.
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Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear function and $b \in \mathbb{R}^m$. We consider the linear equation $Tx = b$ and solve for $x$. We say $Tx = b$ is homogeneous if $b = 0$, and is inhomogeneous otherwise.

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Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear function and $b \in \mathbb{R}^m$. We consider the linear equation $Tx = b$ and solve for $x$. We say $Tx = b$ is \textit{homogeneous} if $b = 0$, and is \textit{inhomogeneous} otherwise.

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**Proof.** For (1) just calculate.
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Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear function and $b \in \mathbb{R}^m$. We consider the linear equation $Tx = b$ and solve for $x$. We say $Tx = b$ is homogeneous if $b = 0$, and is inhomogeneous otherwise.

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Proof. For (1) just calculate.
2) $\implies$ 3) by induction.
Homogeneous equations: abstract view

Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear function and \( b \in \mathbb{R}^m \). We consider the linear equation \( Tx = b \) and solve for \( x \).
We say \( Tx = b \) is homogeneous if \( b = 0 \), and is inhomogeneous otherwise.

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Suppose \( x_1, \ldots, x_r \) are solutions to the homogeneous eqn \( Tx = 0 \).

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Homogeneous equations: abstract view

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear function and $b \in \mathbb{R}^m$. We consider the linear equation $T \mathbf{x} = \mathbf{b}$ and solve for $\mathbf{x}$. We say $T \mathbf{x} = \mathbf{b}$ is homogeneous if $b = 0$, and is inhomogeneous otherwise.

**Proposition**

Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_r$ are solutions to the homogeneous eqn $T \mathbf{x} = 0$.

1. $\mathbf{x} = \mathbf{0}$ is a homogeneous soln, (existence).
2. $\mathbf{x}_1 + \mathbf{x}_2, \lambda \mathbf{x}_1$ are homog solns for any $\lambda \in \mathbb{R}$.
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$$T(\mathbf{x}_1 + \mathbf{x}_2) = T\mathbf{x}_1 + T\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$ 

Similarly, $T(\lambda \mathbf{x}_1) =$
Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear function and \( b \in \mathbb{R}^m \). We consider the \textit{linear equation} \( Tx = b \) and solve for \( x \).
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**Terminology** We express (2) by saying that the set of homogeneous solns is \textit{closed under addition and scalar multiplication}. We express (3) by saying homog solns are \textit{closed under linear combinations}.
Inhomogeneous equations: abstract view

The solutions to an inhomogeneous linear equation \( Ax = b \) are related to the solutions of the corresponding homogeneous equation \( Ax = 0 \), as long as solutions exist!

Proposition

Let \( x = x_p \) be a solution to \( Ax = b \).

Then \( x = x_p + x_h \) is also a solution for any homogeneous solution \( x_h \) to \( Ax = 0 \).

Every solution to \( Ax = b \) has this form.

Remark

The importance of the proposition is that if you know a particular inhomogeneous solution, and the general homogeneous solution, then you know the general inhomogeneous solution too.

Proof.

1) Let \( x_0 \) be a solution to \( Ax = b \).

Let \( x_h = x_0 - x_p \) so \( x_0 = x_p + x_h \).

It suffices to show \( x_h \) is a homogeneous solution:

Corollary

If \( x = 0 \) is the unique homogeneous solution, then an inhomogeneous solution is unique (assuming it exists).

Daniel Chan (based on Ian Doust’s notes) (UNSW)

Chapter 4: Linear Equations

Semester 1 2015
Inhomogeneous equations: abstract view

The solutions to an inhomogeneous linear eqn $Tx = b$ are related to the solns of the corresponding homogeneous eqn $Tx = 0$.

Proposition

Let $x = x_p$ be a soln to $Tx = b$.

Then $x = x_p + x_h$ is also a solution for any homog soln $x_h$ to $Tx = 0$.

Every soln to $Tx = b$ has this form.

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The importance of the proposition is that if you know a particular inhomogeneous soln, and the general homogeneous soln, then you know the general inhomogeneous soln too.

Proof.

1) Let $x_0$ be a soln to $Tx = b$.

Let $x_h = x_0 - x_p$ so $x_0 = x_p + x_h$.

It suffices to show $x_h$ is a homog soln:

Corollary

If $x = 0$ is the unique homogeneous soln, then an inhomogeneous soln is unique (assuming it exists).
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If $x = 0$ is the unique homogeneous soln, then an inhomogeneous soln is unique (assuming it exists).
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The solutions to an inhomogeneous linear eqn $T\mathbf{x} = \mathbf{b}$ are related to the solns of the corresponding homogeneous eqn $T\mathbf{x} = \mathbf{0}$, as long as solns exist!

**Proposition**

Let $\mathbf{x} = \mathbf{x}_p$ be a soln to $T\mathbf{x} = \mathbf{b}$.

Every soln to $T\mathbf{x} = \mathbf{b}$ has this form.

**Remark**

The importance of the proposition is that if you know a particular inhomogeneous soln, and the general homogeneous soln, then you know the general inhomogeneous soln too.

**Proof.**

1) Let $\mathbf{x}_0$ be a soln to $T\mathbf{x} = \mathbf{b}$.

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**Proposition**

Let $x = x_p$ be a soln to $Tx = b$.

1. Then $x = x_p + x_h$ is also a solution for any homog soln $x_h$ to $Tx = 0$. 

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The importance of the proposition is that if you know a particular inhomogeneous soln, and the general homogeneous soln, then you know the general inhomogeneous soln too.

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The solutions to an inhomogeneous linear eqn $T\mathbf{x} = \mathbf{b}$ are related to the solns of the corresponding homogeneous eqn $T\mathbf{x} = \mathbf{0}$, as long as solns exist!

**Proposition**

Let $\mathbf{x} = \mathbf{x}_p$ be a soln to $T\mathbf{x} = \mathbf{b}$.

1. Then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ is also a solution for any homog soln $\mathbf{x}_h$ to $T\mathbf{x} = \mathbf{0}$.
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**Proof.** 1)
The solutions to an inhomogeneous linear eqn \( T \mathbf{x} = \mathbf{b} \) are related to the solns of the corresponding homogeneous eqn \( T \mathbf{x} = \mathbf{0} \), as long as solns exist!

**Proposition**

Let \( \mathbf{x} = \mathbf{x}_p \) be a soln to \( T \mathbf{x} = \mathbf{b} \).

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1) 

2) Let \( \mathbf{x}_0 \) be a soln to \( T \mathbf{x} = \mathbf{b} \).
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**Corollary** If $\mathbf{x} = \mathbf{0}$ is the unique homogeneous soln, then an inhomogeneous soln is unique (assuming it exists).
Relating the abstract view with result from Gaussian elimination

E.g. Consider the linear eqn \( A \mathbf{x} = \mathbf{b} \) corresponding to the reduced row echelon augmented matrix

\[
\begin{bmatrix}
1 & 2 & 0 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Assigning parameters \( x_2 = \lambda, \ x_4 = \mu \), back-substitution gives the general soln as

\[
\mathbf{x} = \begin{bmatrix}
8 \\
0 \\
3 \\
0
\end{bmatrix} + \lambda \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix} + \mu \begin{bmatrix}
-4 \\
0 \\
-2 \\
1
\end{bmatrix}.
\]

Setting \( \lambda = \mu = 0 \) gives the particular soln

\[
\begin{bmatrix}
8 \\
0 \\
3 \\
0
\end{bmatrix}
\]

and the proposition last slide = \( \Rightarrow \) the general homog soln is the span of Daniel Chan (based on Ian Doust’s notes) (UNSW)
E.g. Consider the linear eqn $Ax = b$ corresponding to the reduced row echelon augmented matrix

$$
\begin{bmatrix}
1 & 2 & 0 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

Assigning parameters $x_2 = \lambda$, $x_4 = \mu$, back-substitution gives the general soln as

$$
\begin{bmatrix}
8 \\
0 \\
3 \\
0
\end{bmatrix}
+ \lambda \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix}
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-4 \\
0 \\
-2 \\
1
\end{bmatrix}
$$

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$$
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8 \\
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Relating the abstract view with result from Gaussian elimination

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$$
\begin{pmatrix}
8 \\
0 \\
3 \\
0
\end{pmatrix} + \lambda
\begin{pmatrix}
-2 \\
1 \\
0 \\
0
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\end{pmatrix} + \lambda \begin{pmatrix}
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1 \\
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$$
\begin{pmatrix}
1 & 2 & 0 & 4 & | & 8 \\
0 & 0 & 1 & 2 & | & 3 \\
0 & 0 & 0 & 0 & | & 0
\end{pmatrix}.
$$

Assigning parameters $x_2 = \lambda, x_4 = \mu$, back-substitution gives the general soln as

$$
x = 
\begin{pmatrix}
8 \\
0 \\
3 \\
0
\end{pmatrix} + 
\lambda 
\begin{pmatrix}
-2 \\
1 \\
0 \\
0
\end{pmatrix} + 
\mu 
\begin{pmatrix}
-4 \\
0 \\
-2 \\
1
\end{pmatrix}.
$$

Setting $\lambda = \mu = 0$ gives the particular soln

$$
\begin{pmatrix}
8 \\
0 \\
3 \\
0
\end{pmatrix}
$$

and the proposition last slide $\Rightarrow$ the general homog soln is the span of
E.g. Consider the linear eqn $A\mathbf{x} = \mathbf{b}$ corresponding to the reduced row echelon augmented matrix

$$
\begin{pmatrix}
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0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Assigning parameters $x_2 = \lambda, x_4 = \mu$, back-substitution gives the general soln as

$$
\mathbf{x} =
\begin{pmatrix}
8 \\
0 \\
3 \\
0
\end{pmatrix} + \lambda \begin{pmatrix}
-2 \\
1 \\
0 \\
0
\end{pmatrix} + \mu \begin{pmatrix}
-4 \\
0 \\
-2 \\
1
\end{pmatrix}.
$$

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$$
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Relating the abstract view with result from Gaussian elimination

**E.g.** Consider the linear eqn $Ax = b$ corresponding to the reduced row echelon augmented matrix

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\end{pmatrix}
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\[
x = \begin{pmatrix} 8 \\ 0 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \end{pmatrix}.
\]

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the general homog soln is the span of
Question

Are there any polynomials \( p(x) = a_0 + a_1 x + \ldots + a_n x^n \) which satisfy the differential eqn

\[
p''(x) + 3x p'(x) - p(x) = x + 2.
\]

The two sides can only be equal if all the coefficients match.

Now \( xp'(x) = a_1 x + 2a_2 x^2 + 3a_3 x^3 + \ldots + na_n x^n \)

\( p''(x) = 2a_2 + 6a_3 x + 12a_4 x^3 + \ldots + n(n-1)a_n x^{n-2} \).
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\]
Application cont’d

Coeff of 1: \[2a_2 + 3 \times 0 - a_0 = 2\]
Coeff of \(x\): \[6a_3 + 3a_1 - a_1 = 1\]
\[\vdots\]
Coeff of \(x^n\): \[0 + 3na_n - a_n = 0.\]
In augmented matrix form, for the variables $a_0, \ldots, a_n$:

\[
\begin{pmatrix}
-1 & 0 & 2 & 0 & \ldots & 0 & 2 \\
0 & 2 & 0 & 6 & & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & & & & & 3n - 1 & 0
\end{pmatrix}
\]
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which clearly has a unique solution as all columns except the last are leading.

**Remark** Note that uniqueness of the soln is independent of the entries in the last column, which matches up with our result relating homogeneous and inhomogeneous solns.
Modern economic problem

Keira's boat company produces 3 types of boats: 1) 1 person, 2) 2 person and 3) 4 person. The company has 3 employees which deal with a) cutting, b) assembly and c) packaging.

The time required to produce each type of boat is given below:

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The cutter will only work 27 hours a week, the assembler 22 and the packager 9.

If Keira is to operate at full capacity, how many of each type of boat will be produced?

Let $x_1$, $x_2$, $x_3$ be the number of 1 person, 2 person and 4 person boats produced a week.

We have one equation for each employer.
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