### Question

Dylan sets off in his rowboat pointing northwest at 2km/h. The current is pushing him south at 0.5km/h. What direction does he end up moving in? What if there’s also wind pushing him at .1km/h east? What’s his speed?

The velocities in this problem are examples of what physicists call *vector quantities*, because they have both a *size* or *magnitude*, and a *direction*.

To answer this question, we need to know how these vector quantities combine to give a final answer.

This is the resultant velocity, which comes from ‘adding’ the vectors, head-to-tail.
The problem here is that to answer the original questions, one has to do a rather painful trigonometry problem!

For problems in 3 dimensions things are even worse!

In this chapter we’ll look at

- How to make these problems easier by working with the $x$, $y$ and $z$ components separately.
- How we can use these ideas to work with problems with more than 3 variables.
- How geometric notions such as distance, angles, lines and planes can be defined when you are working with 27 variables instead of 3!
A (geometric) vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. We usually denote a vector in boldface $\mathbf{v}$ or with an arrow above $\vec{v}$, or a squiggle under $\vec{v}$.

### Equality of vectors

We say that vectors $\mathbf{u}$ and $\mathbf{v}$ are equal, and we write $\mathbf{u} = \mathbf{v}$, if $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude and direction.

**E.g.**

We denote the magnitude of $\mathbf{v}$ by $|\mathbf{v}|$. (There isn’t any notation for the direction!) The zero vector, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.
Vector addition

Given two vectors $u$ and $v$ we can ‘add’ them to produce a new vector $u + v$ as follows:

Adding the zero vector $0$ does nothing as $v + 0 = 0 + v = v$ for all vectors $v$.

There are many physical interpretations of this addition. E.g. 3-way tug-o-war

We let $-v$ denote the vector with the same magnitude but the opposite direction so $v + (-v) = 0$. Note $v + (-v) = 0$.

We define subtraction by $u - v := u + (-v)$. 
Reminder on real numbers

Primary and high school arithmetic uses numbers that we call \textit{real}. For example, the numbers

\[ 0, \ 73, \ -2\frac{1}{5}, \ \sqrt{2}, \ \pi - e^2, \ldots \]

and so on are real numbers.

We will denote the real number system by $\mathbb{R}$. We visualize $\mathbb{R}$ as an infinitely long line:

Within the set $\mathbb{R}$ we have

- Natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
- Integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- Rational numbers, $\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, \ q \neq 0 \right\}$.
- Positive numbers, $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.
- Irrational numbers, i.e. those that aren’t in $\mathbb{Q}$.

and many other subsets of numbers. Sometimes, we refer to numbers as \textit{scalars} (as opposed to vectors).
There is another operation we can perform on vectors, *scalar multiplication*. If $\lambda \in \mathbb{R}$ is a non-negative scalar and $v$ a vector, then $\lambda v$ denotes the vector with the same direction, but magnitude is scaled by $\lambda$, i.e. $|\lambda v| = \lambda |v|$.

If $\lambda < 0$ then we define $\lambda v := |\lambda|(-v)$.

**E.g.** $(-1)v = ??$

If $u = \lambda v$ for some $\lambda \neq 0$, then we say that $u$ and $v$ are **parallel**.

**Q** What vector is parallel to every other vector?
Question

In what sense is vector addition a type of “addition”?

A You can manipulate vector sums much as you can numbers because of the

**Commutative law** \( \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \) (Why?)

**Associative law** \( (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \) (Why?)
Question

In what sense is scalar multiplication a type of “multiplication”?

**Associative law**  \(\lambda(\mu \mathbf{v}) = (\lambda \mu)\mathbf{v}\) (Why?)

**Distributive law**  \(\lambda(\mathbf{v} + \mathbf{w}) = \lambda \mathbf{v} + \lambda \mathbf{w}\) (Vector)  \((\lambda + \mu)\mathbf{v} = \lambda \mathbf{v} + \mu \mathbf{v}\) (Scalar)
**E.g.** Simplify $5(2\mathbf{v} - \mathbf{w}) + 3\mathbf{w}$

**Note** To perform this arithmetic, we only needed the basic properties of vector addition and scalar multiplication above. In general, we will meet lots of contexts where we have a vector addition and scalar multiplication satisfying these axioms. These sets will be called vector spaces. In these cases, we can perform arithmetic as above.
Co-ordinates

To compute with given vectors, we need to introduce co-ordinates on the plane (or space).

The first step is to decide on an ‘x-direction’ and a ‘y-direction’. This is done by choosing a pair of vectors \( \mathbf{i} \) and \( \mathbf{j} \) which have length 1 and which are at right angles to one another.

(The convention is to choose \( \mathbf{j} \) at \( \pi/2 \) anticlockwise from \( \mathbf{i} \).)

### Fact-Definition

Every geometric vector \( \mathbf{a} \) in the plane can be written as \( \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} \) for some unique pair of numbers \( a_1, a_2 \in \mathbb{R} \).

The co-ordinates of \( \mathbf{a} \) (with respect to \( \mathbf{i}, \mathbf{j} \)) is \( \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \) and we express this in notation sometimes as \( \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \).
Co-ordinate arithmetic reflects vector arithmetic

Note that

\[
\begin{pmatrix}
a_1 \\
a_2 \\
\end{pmatrix} + \begin{pmatrix}
b_1 \\
b_2 \\
\end{pmatrix} = (a_1 \mathbf{i} + a_2 \mathbf{j}) + (b_1 \mathbf{i} + b_2 \mathbf{j})
\]

\[
= (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} \quad \text{(Why?)}
\]

\[
= \begin{pmatrix}
a_1 + b_1 \\
a_2 + b_2 \\
\end{pmatrix}.
\]

\[
\lambda \begin{pmatrix}
a_1 \\
a_2 \\
\end{pmatrix} = \lambda (a_1 \mathbf{i} + a_2 \mathbf{j})
\]

\[
= \lambda a_1 \mathbf{i} + \lambda a_2 \mathbf{j}
\]

\[
= \begin{pmatrix}
\lambda a_1 \\
\lambda a_2 \\
\end{pmatrix}.
\]

This says to sum or multiply vectors, we need only sum or multiply co-ordinates.
Example

Question

I walk 5km due west, then 3km north-east, then 6km on a bearing 60° east of north. Where do I end up?

Solution. Take $i$ pointing east and $j$ pointing north.
You can do all this with 3-dimensional geometric vectors too. Here you need basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. You can write each
\[ \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \]
or as $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$. Again
\[
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}, \quad \lambda \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{pmatrix}.
\]
The space $\mathbb{R}^n$

The set of coordinate vectors for geometric vectors in the plane is really just an object you have met already. It is the set of all pairs of real numbers, although here we have written them as a column $\begin{pmatrix} x \\ y \end{pmatrix}$ rather than $(x, y)$.

The set coordinate vectors for geometric vectors in space is just the set of all triples of real numbers.

More generally, let $n$ be a positive integer. An $n$-tuple or $n$-vector is an ordered list of $n$ numbers $a_1, a_2, \ldots, a_n$, written as either a column vector or a row vector:

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad \text{or} \quad a = (a_1, a_2, \ldots, a_n).$$

The set of all $n$-tuples is denoted $\mathbb{R}^n$. Thus

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \\ \pi \end{pmatrix} \in \mathbb{R}^4.$$
We can define vector addition and scalar multiplication on $\mathbb{R}^n$ co-ordinatewise as we saw for co-ordinate vectors:

\[
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{pmatrix} + \begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_n
\end{pmatrix} := \begin{pmatrix}
a_1 + b_1 \\
a_2 + b_2 \\
\vdots \\
a_n + a_n
\end{pmatrix}, \quad \lambda \begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{pmatrix} := \begin{pmatrix}
\lambda a_1 \\
\lambda a_2 \\
\vdots \\
\lambda a_n
\end{pmatrix}.
\]

The ‘zero element’ is $0 = \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}$ and the negative is given by $- \begin{pmatrix}
a_1 \\
\vdots \\
a_n
\end{pmatrix} = \begin{pmatrix}
-a_1 \\
\vdots \\
-a_n
\end{pmatrix}$ (so we can define subtraction!).

Note: each $\mathbb{R}^n$ is a separate system. You can’t add a 3-tuple to a 7-tuple!

Q Compute $3 \begin{pmatrix}
1 \\
1 \\
-2
\end{pmatrix} - 2 \begin{pmatrix}
1 \\
1 \\
-2
\end{pmatrix} = \begin{pmatrix}
\text{expression}
\end{pmatrix}$
Properties of arithmetic on $\mathbb{R}^n$

This ‘vector addition and scalar multiplication inherit good properties from addition and multiplication on $\mathbb{R}$. That is, if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$ then

- **Commutative:** $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$,
- **Associative:** $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$,
- **Distributive:** $\lambda(\mathbf{a} + \mathbf{b}) = \lambda \mathbf{a} + \lambda \mathbf{b}$,
- **Distributive:** $(\lambda + \mu)\mathbf{a} = \lambda \mathbf{a} + \mu \mathbf{a}$.
- **Cancellation:** $\lambda \mathbf{a} = \mathbf{0}$ if and only if $\lambda = 0$ or $\mathbf{a} = \mathbf{0}$,
- etc

**Moral:** You can do algebra in $\mathbb{R}^n$ without running into any problems!

**Proof** Easy from definitions but take up space e.g.
The geometry of $\mathbb{R}^n$

We now have two ways of visualising a 2-tuple $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$:

- As a geometric vector with this coordinate vector,
- As the position of a point in the $xy$ plane.

Sometimes one is more useful; sometimes the other:

- It is more natural to add geometric vectors than positions.
- It is natural to think about angles between geometric vectors.
- It is more natural to think about the distance between points.
Of course the two ways are closely connected. If $A$ is a point with coordinates $(x, y)$ and $O$ is the origin (with position $(0, 0)$), then the geometric vector $\overrightarrow{OA}$ has coordinate vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

More generally, if $A$ and $B$ are points in the plane with coordinates $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then the displacement vector $\overrightarrow{AB} =$

Note $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ so has co-ord vector
Simple geometric applications of vectors

**E.g.** Are \( A = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, B = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, C = \begin{pmatrix} 6 \\ -2 \\ 5 \end{pmatrix} \) collinear?

**E.g.** Are \( A = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, C = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}, D = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \) the vertices of a parallelogram?
Length of vectors in $\mathbb{R}^n$

Many geometric notions in $\mathbb{R}^2$ and $\mathbb{R}^3$ generalise to $\mathbb{R}^n$. Let $a \in \mathbb{R}^n$. The **length** of $a$ is defined to be

$$|a| = \sqrt{a_1^2 + \cdots + a_n^2}.$$ 

In $\mathbb{R}^2$, $|a|$ is the length of the corresponding geometric vector, and also the distance of the point (with coordinates) $a$ from the origin in the $xy$ plane.

**E.g.** What is $|\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}|$?
In general, if \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \), the **distance** between \( \mathbf{a} \) and \( \mathbf{b} \) is defined to be

\[
\text{dist}(\mathbf{a}, \mathbf{b}) = |\mathbf{b} - \mathbf{a}|.
\]

**Example.** Suppose that the point \( A \) has coordinates \((1, 2, 3)\) and the point \( B \) has coordinates \((-1, 2, 5)\). What is the distance between \( A \) and \( B \)?
Lines in $\mathbb{R}^n$

Using our 2 and 3-dim intuition, a line $L$ in $\mathbb{R}^n$ should be determined by

- a point $A$ on the line, say with $a = \overrightarrow{OA}$ and,
- a direction, say given by a non-zero vector $v$

Let’s determine what the general point of $L$ ought to be:

$$x = a + \lambda v, \quad \lambda \in \mathbb{R}$$

This is called the **parametric vector form** of the line $L$. We call $\lambda$ the parameter, and as it varies over $\mathbb{R}$, the variable $x$ varies over all the points of the line $L$.

**Definition**

A *line* in $\mathbb{R}^n$ is any set of the form

$$\{ x \in \mathbb{R}^n \mid x = a + \lambda v, \quad \lambda \in \mathbb{R} \}$$

for some vectors $0 \neq v, a \in \mathbb{R}^n$. Note $a$ gives a point on the line and $v$ its direction.
See MAPLE file
Let \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \) be the coordinate vectors for the points \( A, B \). The midpoint of \( AB \) has coordinates

\[
\overrightarrow{OA} + \frac{1}{2} \overrightarrow{AB} = \mathbf{a} + \frac{1}{2} (\mathbf{b} - \mathbf{a}) = \frac{1}{2} (\mathbf{a} + \mathbf{b}).
\]

Why?

**Q** Show that the diagonals of a parallelogram bisect each other.
In high school, you express a line in the plane in Cartesian form \( ax + by = c \).

**Question**

Find a parametric vector form for the line \( y = 3x + 2 \) in \( \mathbb{R}^2 \).

**A1** Consider 2 points on the line

**A2** We introduce the parameter \( \lambda = \)

**N.B.** There are many other solutions! (What are they?)
Write the line $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ in Cartesian form.

The secret is to eliminate the extra variable $\lambda$!

**Solution.** Write $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ -1 + \lambda \end{pmatrix}$. Then

$$\lambda =$$

What about $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$?
Cartesian form for lines and planes in $\mathbb{R}^3$

Recall that a plane in $xyz$-space can be described by an equation

$$ax + by + cz = d$$

where not all $a, b, c$ are $0$. This is called the *Cartesian form* for the plane. The terms in this equation can of course be re-arranged many ways (see below).

To obtain the cartesian form for a line $L$, we need 2 such equations. Each defines a plane $P_1, P_2$ and solving simultaneously gives the solution $P_1 \cap P_2$. This will be a line unless . . . .

Usually, (but not always) we can write the 2 equations in the form

$$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}$$

for some constants $a_i, v_i$. 
Question

Find the parametric form for \( \frac{x - 2}{3} = \frac{y + 1}{6} = \frac{z - 3}{-2} \).

**A** We can find a point on the line and a direction vector or just introduce the parameter.
Question

Find a cartesian form for

\[ \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R} \]

What about \( \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} \)?
The Cartesian form expresses a line or plane as the solutions to some equations. (Top down)

The parametric form expresses the line or plane by a sophisticated way of listing elements, where running through the list is by letting the parameter range over a set. (Bottom up)

In mathematics, these are the two usual general ways to describe any set.

Both have their uses.

For lines, the parametric form is closest to our geometric picture of the line.
Why bother defining lines in $\mathbb{R}^n$?

- Suppose we are solving equations in $n$ unknowns $x_1, \ldots, x_n$. If $n = 3$, it is often good to visualise the solution set in $x_1x_2x_3$-space.
- For example, solving simultaneously

  \[ a_1x_1 + a_2x_2 + a_3x_3 = a, \quad b_1x_1 + b_2x_2 + b_3x_3 = b \]

  should on geometric grounds, give either a line, plane or the empty set.
- In particular, you can’t get a point or two points etc.
- If $n > 3$, we can use our geometric intuition to understand solutions to many equations provided we generalise our notions of things like lines in $\mathbb{R}^3$ to lines in higher dimensions.
“Directions” of a plane

Thought experiment

What are the “directions” of a plane \( P \subset \mathbb{R}^3 \)?

Since we are interested in directions only, let’s suppose \( P \) passes through \( O \) and that \( \mathbf{v}, \mathbf{w} \in P \) are not parallel.

Hence

Fact

Any vector parallel to \( P \) has the form \( \lambda \mathbf{v} + \mu \mathbf{w} \) for some \( \lambda, \mu \in \mathbb{R} \).

i.e. all other “directions” of \( P \) can be obtained from \( \mathbf{v}, \mathbf{w} \) by combining them using vector operations.
More generally, we consider

**Definition**

Suppose that $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$. A **linear combination** of these vectors is a vector of the form

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots \lambda_k v_k$$

with $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$.

Is \[
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
\]
a linear combination of \[
\begin{pmatrix}
-1 \\
0 \\
-1
\end{pmatrix}
\] and \[
\begin{pmatrix}
2 \\
1 \\
0
\end{pmatrix}
\]?
Span

**Definition**

Let $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$. The span of $v_1, v_2, \ldots, v_k$, written $\text{span}(v_1, \ldots, v_k)$ is the set of all linear combinations of $v_1, v_2, \ldots, v_k$. i.e.

$$\text{span}(v_1, \ldots, v_k) = \{ \lambda_1 v_1 + \lambda_2 v_2 + \ldots \lambda_k v_k | \lambda_1, \ldots, \lambda_k \in \mathbb{R} \}.$$

**Ex.** Describe $\text{span} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

**Ex.** Describe $\text{span} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. 

Daniel Chan (based on Ian Doust’s notes) (UNSW) 
Chapter 1: Introduction to Vectors 
Semester 1 2015
Planes in $\mathbb{R}^n$

**Definition**

A **plane in** $\mathbb{R}^n$ is defined to be a set of the form

$$S = \{ a + \lambda_1 v_1 + \lambda_2 v_2 | \lambda_1, \lambda_2 \in \mathbb{R} \},$$

where $a$, $v_1$ and $v_2$ are fixed vectors in $\mathbb{R}^n$, and $v_1$ and $v_2$ are not parallel.

The expression $x = a + \lambda_1 v_1 + \lambda_2 v_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$ is a parametric vector form for the plane through $a$ parallel to the vectors $v_1$ and $v_2$.

The above picture shows that when $n = 3$, our definition agrees with our old one.
Question

Find a parametric vector equation for the plane through the points \( \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \)

\( \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \) and \( \mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}. \)
Question

Find the Cartesian equation of the plane in $\mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$
In \( \mathbb{R}^n \), the vector \( e_j \) is the \( n \)-tuple with 1 in the \( j \)th position and zeros elsewhere.

\[ \mathbb{R}^2: \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

\[ \mathbb{R}^3: \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]

Obviously, every vector in \( \mathbb{R}^n \) can be written uniquely as a linear combination of \( e_1, \ldots, e_n \), eg

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3.
\]

The vectors \( e_1, \ldots, e_n \) are called the **standard basis vectors** for \( \mathbb{R}^n \).