Chapter 1: Introduction to Vectors (based on Ian Doust’s notes)

Daniel Chan

UNSW

Semester 1 2017
A typical problem

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Q What if the Millenium Falcon is in the Death Star which is moving ...?
Goals of this chapter

Note that to answer the question above, you need to know both the magnitudes $5\text{ms}^{-1}$, $10\text{ms}^{-1}$ and the directions of motion.

In this chapter we'll intro new mathematical objects called vectors which encode info about magnitudes & directions.

Note real numbers only encode magnitudes and sign (i.e. $+$ or $-$).

study how vectors combine to answer questions such as that above.

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Equality of vectors

We say vectors $\mathbf{u}$ & $\mathbf{v}$ are equal, and write $\mathbf{u} = \mathbf{v}$, if $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude and direction.
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The zero vector, denoted by $0$, has length 0. It is the only vector with no specific direction.
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Adding the zero vector $\mathbf{0}$ does nothing as $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$. 

There are many physical interpretations of this addition. E.g. 3-way tug-o-war. We let $-\mathbf{v}$ denote the vector with the same magnitude but the opposite direction so $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$. We define subtraction by $\mathbf{u} - \mathbf{v} := \mathbf{u} + (-\mathbf{v})$. 

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Vector addition

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Reminder on real numbers

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\[ 0, 73, -2\frac{1}{5}, \sqrt{2}, \pi - e^2, \ldots \]

and so on are real numbers.
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Scalar multiplication

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If $\lambda < 0$ then we define $\lambda \mathbf{v} := |\lambda| (\mathbf{v})$.

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If \( \mathbf{u} = \lambda \mathbf{v} \), then we say that \( \mathbf{u} \) and \( \mathbf{v} \) are *parallel*. 
Properties of vector addition

Question

In what sense is vector addition a type of “addition”?

A you can manipulate vector sums much as you can numbers because of the Commutative law $v + w = w + v$ (Why?)

Associative law $\left( u + v \right) + w = u + \left( v + w \right)$ (Why?)

Since these sums equal, we’ll write it simply as $u + v + w$.

Similarly, we may omit brackets when adding 4 or more vectors together.

Challenge Q Why?
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Associative law

\[ \lambda (\mu v) = (\lambda \mu) v \]

(Why?)

Distributive law

\[ \lambda (v + w) = \lambda v + \lambda w \] (Vector)

\[ (\lambda + \mu) v = \lambda v + \mu v \] (Scalar)
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- $\lambda(v + w) = \lambda v + \lambda w$ (Vector)
- $(\lambda + \mu)v = \lambda v + \mu v$ (Scalar)
**E.g.** Let \( w = u + 2v \). Are \( 2w - 4v \) \& \( u \) parallel?

We simplify \( 2w - 4v = 2(u + 2v) - 4v \).

Note To perform this arithmetic, we only needed the basic properties of vector addition and scalar multiplication above. In general, we will meet lots of contexts where we have a vector addition and scalar multiplication satisfying these axioms. These sets will be called vector spaces. In these cases, we can perform arithmetic as above.
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Co-ordinates

To put co-ordinates on the plane we need to specify:

- an origin point $O$ where (co-ord axes cross), and
- a pair of vectors $i$ and $j$ which have length 1 and which are at right angles to one another. (The convention is to choose $j$ at $\pi/2$ anticlockwise from $i$.)

So $i, j$ give direction of coord axes & scale.

Fact-Definition

Every geometric vector $a$ in the plane can be written as $a = a_1 i + a_2 j$ for some unique pair of numbers $a_1, a_2 \in \mathbb{R}$.

The co-ordinates or coordinate vector of $a$ (with respect to $i, j$) is $(a_1, a_2)$.
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The co-ordinates or coordinate vector of $a$ (with respect to $i, j$) is $\left( \begin{array}{c} a_1 \\ a_2 \end{array} \right)$.
Co-ordinate arithmetic reflects vector arithmetic

Let \( \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j}, \) \( \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} \) so coords are \((a_1, a_2), (b_1, b_2)\).
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Let \( \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} \), \( \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} \) so coords are \( \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \), \( \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \).

The coords of

\[
\mathbf{a} \pm \mathbf{b} = (a_1 \mathbf{i} + a_2 \mathbf{j}) \pm (b_1 \mathbf{i} + b_2 \mathbf{j}) = (a_1 \pm b_1)\mathbf{i} + (a_2 \pm b_2)\mathbf{j}
\]

are \( \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \end{pmatrix} \).

Sim, the coords of

\[
\lambda \mathbf{a} = \lambda(a_1 \mathbf{i} + a_2 \mathbf{j}) = \lambda a_1 \mathbf{i} + \lambda a_2 \mathbf{j}
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**Upshot** This says to sum, subtract or multiply vectors, we need only sum, subtract or multiply coords.
Example

To find coords recall given a right angle triangle,
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Question

I walk 1km due west, then 4km on a bearing $30^\circ$ east of north. Where do I end up?
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Solution. Take $i$ pointing east and $j$ pointing north & units are km.
You can do all this with 3-dimensional geometric vectors too.
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$$
You can do all this with 3-dimensional geometric vectors too. Here you need basis vectors $i$, $j$ and $k$. You can write vectors in form

$$a = a_1 i + a_2 j + a_3 k, \quad b = b_1 i + b_2 j + b_3 k$$

which have coords 

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$
3-dimensional version

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$$a = a_1i + a_2j + a_3k, \quad b = b_1i + b_2j + b_3k$$

which have coords $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Again

$\text{coords of } a + b = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$, $\text{coords of } \lambda a = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{pmatrix}$. 
We can generalise coordinate vectors to any number of components!
We can generalise coordinate vectors to any number of components! Let $n$ be a positive integer. An $n$-tuple or $n$-vector is an ordered list of $n$ numbers $a_1, a_2, \ldots, a_n$, written as either a column vector or (less often in this course) a row vector:

$$
\mathbf{a} = \begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{pmatrix}
$$
or

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\mathbf{a} = (a_1, a_2, \ldots, a_n).
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The set of all $n$-tuples is denoted $\mathbb{R}^n$. Thus

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \\ \pi \end{pmatrix} \in \mathbb{R}^4.$$
The space $\mathbb{R}^n$

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Hence coord vectors of 3-dim vectors lie in $\mathbb{R}^3$ whilst those of 2-dim vectors lie in $\mathbb{R}^2$. 
We can define vector addition and scalar multiplication on $\mathbb{R}^n$ coordinatewise as we saw for coordinate vectors:

$$
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{pmatrix}
+ 
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_n
\end{pmatrix} := 
\begin{pmatrix}
a_1 + b_1 \\
a_2 + b_2 \\
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\end{pmatrix},
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a_n
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\lambda a_1 \\
\lambda a_2 \\
\vdots \\
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$$

The 'zero element' is $0 = \begin{pmatrix}
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0 \\
\vdots \\
0
\end{pmatrix}$ and the negative is given by $- \begin{pmatrix}
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\vdots \\
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-a_n
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Note: each $\mathbb{R}^n$ is a separate system. You can't add a 3-tuple to a 7-tuple!

Compute $3 \begin{pmatrix}
1 \\
1 \\
-2
\end{pmatrix} - 2 \begin{pmatrix}
1 \\
1 \\
-2
\end{pmatrix} =$
We can define vector addition and scalar multiplication on $\mathbb{R}^n$ coordinatewise as we saw for coordinate vectors:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} := \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + a_n \end{pmatrix}, \quad \lambda \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} := \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}.$$
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Arithmetic on $\mathbb{R}^n$

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\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n \\
\end{pmatrix} + \begin{pmatrix}
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b_2 \\
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b_n \\
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**Q** Compute $3 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} =$
This ‘vector addition and scalar multiplication inherit good properties from addition and multiplication on \( \mathbb{R} \). That is, if \( a, b, c \in \mathbb{R}^n \) and \( \lambda, \mu \in \mathbb{R} \) then

- **Commutative:** \( a + b = b + a \),
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**Moral:** You can do algebra in \( \mathbb{R}^n \) without running into any problems!
Properties of arithmetic on $\mathbb{R}^n$

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**Moral:** You can do algebra in $\mathbb{R}^n$ without running into any problems!

**Proof** Easy from definitions but take up space e.g.
Displacement vector

Put coords on the plane by specifying $O, \textbf{i}, \textbf{j}$ as usual.
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Put coords on the plane by specifying $O, \mathbf{i}, \mathbf{j}$ as usual.

Given any 2 points $A, B$ on the plane, we define the displacement vector $\overrightarrow{AB}$ to be the geometric vector with tail $A$ & head $B$ i.e.

\[ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \]

**Important Remark**

In high school you would have considered coords of the point $A$ (as opposed to a vector).

\[ \text{coords} A = \text{coords} \overrightarrow{OA} \]

\[ \text{coords} \overrightarrow{AB} = \text{coords} B - \text{coords} A \]

$\overrightarrow{OA}$ is called the position vector of $A$ (with respect to $O$).

These observations also hold if the points are in space with coords specified.
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Simple geometric applications of vectors

E.g. Are \( A = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, B = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, C = \begin{pmatrix} 6 \\ -2 \\ 5 \end{pmatrix} \) collinear?

E.g. Are \( A = (0, 1), B = (1, 3), C = (4, 5), D = (3, 3) \) the vertices of a parallelogram?
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Length and distance in $\mathbb{R}^n$

Pythagoras’ thm $\implies$ the length of a geometric vector with coords $\left(\begin{array}{c} a_1 \\ a_2 \end{array}\right)$ is $\sqrt{a_1^2 + a_2^2}$. This suggests the following generalisation of the length concept to $\mathbb{R}^n$.

**Definition**

Let $a, b \in \mathbb{R}^n$.

The length of $a$ is defined to be $|a| = \sqrt{a_1^2 + \cdots + a_n^2}$.

The distance between $a$ and $b$ is defined to be $\text{dist}(a, b) = |b - a|$.

**Example.**

a) What is $\left|\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right|$?

b) Suppose that the point $A$ has coordinates $(1, 2, 3)$ and the point $B$ has coordinates $(-1, 2, 5)$. What is the distance between $A$ and $B$?
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Length and distance in $\mathbb{R}^n$

Pythagoras’ thm $\implies$ the length of a geometric vector with coords \(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\) is $\sqrt{a_1^2 + a_2^2}$. This suggests the following generalisation of the length concept to $\mathbb{R}^n$.

**Definition**

Let $a, b \in \mathbb{R}^n$.

- The *length* of $a$ is defined to be $|a| = \sqrt{a_1^2 + \cdots + a_n^2}$.
- the *distance* between $a$ and $b$ is defined to be $\text{dist}(a, b) = |b - a|$.
Pythagoras’ thm \(\Rightarrow\) the length of a geometric vector with coords \(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\) is \(\sqrt{a_1^2 + a_2^2}\). This suggests the following generalisation of the length concept to \(\mathbb{R}^n\).

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**Example.** a) What is \(|\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}|\)?
Length and distance in \( \mathbb{R}^n \)

Pythagoras’ thm \( \Rightarrow \) the length of a geometric vector with coords \( \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \) is \( \sqrt{a_1^2 + a_2^2} \). This suggests the following generalisation of the length concept to \( \mathbb{R}^n \).

**Definition**

Let \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \).
- The *length* of \( \mathbf{a} \) is defined to be \( |\mathbf{a}| = \sqrt{a_1^2 + \cdots + a_n^2} \).
- The *distance* between \( \mathbf{a} \) and \( \mathbf{b} \) is defined to be \( \text{dist}(\mathbf{a}, \mathbf{b}) = |\mathbf{b} - \mathbf{a}| \).

**Example.**

a) What is \( \left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right| \)?

b) Suppose that the point \( A \) has coordinates \((1, 2, 3)\) and the point \( B \) has coordinates \((-1, 2, 5)\). What is the distance between \( A \) and \( B \)?
Using our 2 and 3-dim intuition, a line $L$ in $\mathbb{R}^n$ should be determined by

- A point $A$ on the line, say with $\vec{OA}$, and
- A direction, say given by a non-zero vector $v$.

Let's determine what the general point of $L$ ought to be:

$$x = a + \lambda v, \quad \lambda \in \mathbb{R}$$

This is called the **parametric vector form** of the line $L$.

**Definition**

A line in $\mathbb{R}^n$ is any set of the form

$$\{ x \in \mathbb{R}^n | x = a + \lambda v, \lambda \in \mathbb{R} \}$$

for some fixed vectors $0 \neq v, a \in \mathbb{R}^n$.

Note $a$ gives a point on the line and $v$ its direction.
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$$\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \mathbf{a} + \lambda \mathbf{v}, \lambda \in \mathbb{R}\}$$
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Note: $\mathbf{a}$ gives a point on the line and $\mathbf{v}$ its direction.
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Lines in $\mathbb{R}^n$

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Let’s determine what the general point of $L$ ought to be:

$$x = a + \lambda v, \quad \lambda \in \mathbb{R}$$

This is called the *parametric vector form* of the line $L$. We call $\lambda$ the parameter, and as it varies over $\mathbb{R}$, the variable $x$ varies over all the points of the line $L$. 
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**Definition**

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for some fixed vectors $0 \neq v, a \in \mathbb{R}^n$. Note $a$ gives a point on the line and $v$ its direction.
See MAPLE file
Midpoints

Let $a, b \in \mathbb{R}^n$ be the position vectors for the points $A, B$. The midpoint of $AB$ has coordinates $\vec{OA} + \frac{1}{2} \vec{AB} = a + \frac{1}{2} (b - a) = \frac{1}{2} (a + b)$.

Why?

Show that the diagonals of a parallelogram bisect each other.
Let \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \) be the position vectors for the points \( A, B \). The midpoint of \( AB \) has coordinates \( \frac{\mathbf{a} + \mathbf{b}}{2} \).
Let \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \) be the position vectors for the points \( A, B \). The midpoint of \( AB \) has coordinates

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Why?
Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ be the position vectors for the points $A, B$. The midpoint of $AB$ has coordinates

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Why?

**Q** Show that the diagonals of a parallelogram bisect each other.
Finding parametric forms for lines from Cartesian form

In high school, you express a line in the plane in Cartesian form \( ax + by = c \).
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**Question**

Find a parametric vector form for the line $y = 3x + 2$ in $\mathbb{R}^2$. 

N.B. There are many other solutions! (What are they?)
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**A1** Consider 2 points on the line
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**A1** Consider 2 points on the line

**A2** We introduce the parameter \( \lambda = \)
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Question

Write the line \( \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \) in Cartesian form.
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The secret is to eliminate the extra variable \( \lambda \)!
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The secret is to eliminate the extra variable \(\lambda\)!

**Solution.** Write \(\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ -1 + \lambda \end{pmatrix} \).
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\[\lambda = \text{[ unspecified]}\]
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\[ \lambda = \]

What about \( x = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \)?
Cartesian form for lines and planes in $\mathbb{R}^3$

Recall that a plane in $xyz$-space can be described by an equation

$$ax + by + cz = d$$

where not all $a$, $b$, $c$ are 0. This is called the Cartesian form for the plane. The terms in this equation can of course be re-arranged many ways (see below).

To obtain the cartesian form for a line $L$, we need 2 such equations. Each defines a plane $P_1$, $P_2$ and solving simultaneously gives the solution $P_1 \cap P_2$. This will be a line unless . . .

Usually, (but not always) we can write the 2 equations in the form

$$x - a_1 v_1 = y - a_2 v_2 = z - a_3 v_3$$

for some constants $a_i$, $v_i$. 
Recall that a plane in $xyz$-space can be described by an equation

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Usually, (but not always) we can write the 2 equations in the form

$$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}$$

for some constants $a_i, v_i$. 
Question

Find the parametric form for \( \frac{x - 2}{3} = \frac{y + 1}{6} = \frac{z - 3}{-2} \).
Question

Find the parametric form for \( \frac{x - 2}{3} = \frac{y + 1}{6} = \frac{z - 3}{-2} \).

A  We can find a point on the line and a direction vector or just introduce the parameter
Question

Find a cartesian form for

\[ \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R} \]
Question

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What about \( \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} \)?
The Cartesian form expresses a line or plane as the solutions to some equations. (Top down)
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Parametric vs Cartesian form

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- In mathematics, these are the two usual general ways to describe any set.
The Cartesian form expresses a line or plane as the solutions to some equations. (Top down)

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In mathematics, these are the two usual general ways to describe any set.

Both have their uses.

For lines, the parametric form is closest to our geometric picture of the line.
Suppose we are solving equations in \( n \) unknowns \( x_1, \ldots, x_n \). If \( n = 3 \), it is often good to visualise the solution set in \( x_1x_2x_3 \)-space.

For example, solving simultaneously

\[
\begin{align*}
a_1x_1 + a_2x_2 + a_3x_3 &= a, \\
b_1x_1 + b_2x_2 + b_3x_3 &= b
\end{align*}
\]

should on geometric grounds, give either a line, plane or the empty set. In particular, you can't get a point or two points etc. If \( n > 3 \), we can use our geometric intuition to understand solutions to many equations provided we generalise our notions of things like lines in \( \mathbb{R}^3 \) to lines in higher dimensions.
Why bother defining lines in $\mathbb{R}^n$?

- Suppose we are solving equations in $n$ unknowns $x_1, \ldots, x_n$. If $n = 3$, it is often good to visualise the solution set in $x_1x_2x_3$-space.

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Thought experiment

What are the “directions” of a plane $P \subset \mathbb{R}^3$?
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Since we are interested in directions only, let’s suppose \( P \) passes through \( O \) and that \( \mathbf{v}, \mathbf{w} \in P \) are not parallel.
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What are the “directions” of a plane $P \subset \mathbb{R}^3$?

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Hence

Fact

Any vector parallel to $P$ has the form $\lambda v + \mu w$ for some $\lambda, \mu \in \mathbb{R}$. 
Thought experiment

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Fact

Any vector parallel to $P$ has the form $\lambda v + \mu w$ for some $\lambda, \mu \in \mathbb{R}$.

i.e. all other “directions” of $P$ can be obtained from $v, w$ by combining them using vector operations.
More generally, we consider

**Definition**

Suppose that \( v_1, v_2, \ldots, v_k \in \mathbb{R}^n \). A linear combination of these vectors is a vector of the form

\[
\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k
\]

with \( \lambda_1, \ldots, \lambda_k \in \mathbb{R} \).

Is

\[
\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}
\]

a linear combination of

\[
\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}
\]

and

\[
\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}
\]

?
Linear combination

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-1 \\
0 \\
-1
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2 \\
0
\end{pmatrix}?
$$
Definition

Let $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$. The span of $v_1, v_2, \ldots, v_k$, written $\text{span}(v_1, \ldots, v_k)$ is the set of all linear combinations of $v_1, v_2, \ldots, v_k$. 

Example:

Describe $\text{span}(2v)$. More generally, $\text{span}(v)$ is the...
Definition

Let \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n \). The *span* of \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \), written \( \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) \) is the set of all linear combinations of \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \). i.e.

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\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \{ \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k | \lambda_1, \ldots, \lambda_k \in \mathbb{R} \}.
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Definition

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\]

Ex. Describe \( \text{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \).
Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. The span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, written $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. i.e.

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Ex. Describe $\text{span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
Planes in $\mathbb{R}^n$

**Definition**

A *plane in* $\mathbb{R}^n$ *is defined to be a set of the form*

$$S = \{a + \lambda_1 v_1 + \lambda_2 v_2 | \lambda_1, \lambda_2 \in \mathbb{R}\},$$

where $a$, $v_1$, and $v_2$ are fixed vectors in $\mathbb{R}^n$, and $v_1$ and $v_2$ are not parallel.

The expression $x = a + \lambda_1 v_1 + \lambda_2 v_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$ is a parametric vector form for the plane through $a$ parallel to the vectors $v_1$ and $v_2$.

The above picture shows that when $n = 3$, our definition agrees with our old one.

What if $v_1, v_2$ above are parallel?
A **plane in** \( \mathbb{R}^n \) is defined to be a set of the form

\[
S = \{ a + \lambda_1 v_1 + \lambda_2 v_2 | \lambda_1, \lambda_2 \in \mathbb{R} \},
\]

where \( a, v_1 \) and \( v_2 \) are fixed vectors in \( \mathbb{R}^n \), and \( v_1 \) and \( v_2 \) are not parallel.
A plane in $\mathbb{R}^n$ is defined to be a set of the form

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where $a$, $v_1$ and $v_2$ are fixed vectors in $\mathbb{R}^n$, and $v_1$ and $v_2$ are not parallel.

The expression $x = a + \lambda_1 v_1 + \lambda_2 v_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$ is a parametric vector form for the plane through $a$ parallel to the vectors $v_1$ and $v_2$. 
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The above picture shows that when $n = 3$, our definition agrees with our old one.
Planes in $\mathbb{R}^n$

**Definition**

A *plane in* $\mathbb{R}^n$ *is defined to be a set of the form*

$$S = \{ a + \lambda_1 v_1 + \lambda_2 v_2 | \lambda_1, \lambda_2 \in \mathbb{R} \},$$

*where* $a$, $v_1$ and $v_2$ *are fixed vectors in* $\mathbb{R}^n$, *and* $v_1$ and $v_2$ *are not parallel.*

The expression $x = a + \lambda_1 v_1 + \lambda_2 v_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$ *is a parametric vector form for the plane through* $a$ *parallel to the vectors* $v_1$ *and* $v_2$.

The above picture shows that when $n = 3$, our definition agrees with our old one.

**Q** What if $v_1, v_2$ above are parallel?
Question

Find a parametric vector equation for the plane through the points \( \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \)
\( \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \) and \( \mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}. \)
Finding Cartesian form for planes from parametric form

**Question**

Find the Cartesian equation of the plane in $\mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$
Meanwhile back at the Death Star

Question

The Millenium Falcon, at coords \( \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \) is flying in direction \( \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \).
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The Millenium Falcon, at coords \( \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \) is flying in direction \( \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \). Will it hit the Death Star wall, a plane with Cartesian eqn \( x - y - z = 1 \)?
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The Millenium Falcon, at coords \[
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**A** Without the Death Star, the flight trajectory would be the *ray* with parametric equation
In $\mathbb{R}^n$, the vector $e_j$ is the $n$-tuple with 1 in the $j$th position and zeros elsewhere.
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$\mathbb{R}^2$: $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. 

The vectors $e_1, \ldots, e_n$ are called the standard basis vectors for $\mathbb{R}^n$. 
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\( \mathbb{R}^3 \): \( \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \), \( \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \), \( \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \).
Standard basis vectors for $\mathbb{R}^n$

In $\mathbb{R}^n$, the vector $e_j$ is the $n$-tuple with 1 in the $j$th position and zeros elsewhere.

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$\mathbb{R}^3$: $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Obviously, every vector in $\mathbb{R}^n$ can be written uniquely as a linear combination of $e_1, \ldots, e_n$, eg

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3.$$ 

The vectors $e_1, \ldots, e_n$ are called the *standard basis vectors* for $\mathbb{R}^n$.