Lecture 8: Span

Aim Lecture Give parametric description of subspaces.

Definitions

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq V =$ vect space / field $\mathbb{F}$. The span of S is the set of lin combns of S,

$\text{Span}(S) = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$

$= \{\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n \,|\lambda, \ldots, \lambda_n \in \mathbb{F}\}$.

e.g. 1 Span $(\varnothing) = \mathbf{0}$.

e.g. 2 If $\mathbf{v} \in \mathbb{R}^3$ then

$\text{Span}(\mathbf{v}) = \{\lambda \mathbf{v} \,|\lambda \in \mathbb{R}\}$

$=$ line with parameteric form $\mathbf{x} = \lambda \mathbf{v}, \lambda \in \mathbb{R}$.

e.g. 3 Recall for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ non-parallel

$\text{Span}(\mathbf{v}, \mathbf{w}) = \{\lambda \mathbf{v} + \mu \mathbf{w} \,|\lambda, \mu \in \mathbb{R}\}$

$=$ plane containing $\mathbf{v}, \mathbf{w}$.

e.g. 4 In vector space $\mathbb{P}$,

$\text{Span}(1, x, x^2) = \{\lambda_0 1 + \lambda_1 x + \lambda_2 x^2 | \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}$

$= \mathbb{P}_2$, the subspace of

Defn If $V = \text{Span}(S)$ then we say that S spans V or that is a spanning set for V.

e.g. 5 $\mathbb{P}_2$ is spanned by

Spanning sets are subspaces

Thm Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq V =$ vect space / field $\mathbb{F}$. Then i) $\text{Span}(S)$ is a subspace of V

ii) $\text{Span}(S) \supseteq S$.

Proof: i) Check subspace axioms.

a. Zero: $\mathbf{0} =$

b. Closure under addn. Let $\mathbf{v}, \mathbf{v'} \in \text{Span}(S)$ so are lin combns of S i.e.

$\mathbf{v} = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n$

$\mathbf{v}' = \lambda_1' \mathbf{v}_1 + \ldots + \lambda_n' \mathbf{v}_n$

for some scalars $\lambda_1, \ldots, \lambda_n, \lambda_1', \ldots, \lambda_n' \in \mathbb{F}$.

$\mathbf{v} + \mathbf{v}' =$

is a lin combn of S so is in Span(S).

$\therefore$ Span(S) is closed under addn.

c. Closed under scalar multn. For $\lambda \in \mathbb{F}$ and $\mathbf{v}, \mathbf{v}_i, \lambda_i$ as in b),

$\lambda \mathbf{v} =$

So Span(S) is closed under scalar multn.

Closure axioms hold so Span(S) is a subspace of V.

ii) We show $\mathbf{v}_i \in$ Span(S) for every i.

$\mathbf{v}_i =$

so $\mathbf{v}_i \in$ Span(S) & Span$(S) \supseteq S$.

Subspaces of $\mathbb{R}^3$

Prop Any subspace W of $\mathbb{R}^3$ is one of the following:

a) a line through $\mathbf{0}$,

b) a plane containing $\mathbf{0}$,

c) $\mathbb{R}^3$ or d) $\mathbf{0}$.

Proof: If $W \neq \mathbf{0}$ then pick $\mathbf{w}_1 \in W - \mathbf{0}$.

Prop lect 7 $\implies W \supseteq \text{Span}(\mathbf{w}_1)$

so either $W =$ line

or we can find $\mathbf{w}_2 \in W - \text{Span}(\mathbf{w}_1)$. In this case $W \supseteq \text{Span}(\mathbf{w}_1, \mathbf{w}_2)$.

Either $W =$ plane Span

or we can find $\mathbf{w}_3 \in$

In latter case, $W = \mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = \mathbb{R}^3$.

Computing span in $\mathbb{F}^m$ via matrices

$$\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \ldots, \mathbf{a}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \in \mathbb{F}^m$$

Let $A =$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ i.e.

$A = (\mathbf{a}_1\, \mathbf{a}_2 \ldots \mathbf{a}_n)$.

Propn The lin combn

$x_1\, \mathbf{a}_1 + \ldots + x_n\, \mathbf{a}_n = A\, \mathbf{x}$

Proof:

$$A\,\mathbf{x} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Corollary $\mathrm{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$ consists of vectors $\mathbf{b}$ such that $A\,\mathbf{x} = \mathbf{b}$ has a soln.

Defn For an $m \times n$-matrix A as above, the column space of A, denoted $\mathrm{col}(A)$ is

$\mathrm{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$.

Stupid e.g. 6 Consider what are called the standard basis vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{F}^m$$

Corresponding matrix is $A = I$. You can always solve $I\,\mathbf{x} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{F}^m$.

In fact a soln is $\mathbf{x} =$

Thus any $\mathbf{b}$ lies in $\text{Span}(\mathbf{e}_1, \ldots, \mathbf{e}_m)$.

$\therefore$ $\text{Span}(\mathbf{e}_1, \ldots, \mathbf{e}_m) = \mathbb{F}^m$, or, in other words $\mathbf{e}_1, \ldots, \mathbf{e}_m$ span $\mathbb{F}^m$.

e.g. 7 Is $\mathbf{b} = (0, 1, 2)^T$ in

$\text{Span}((1, 2, 3)^T, (4, 5, 6)^T)$?

i.e. Can you solve $x_1(1, 2, 3)^T + x_2(4, 5, 6)^T = \mathbf{b}$?

OR can you solve $A\mathbf{x} = \mathbf{b}$ with $A =$

Hence we can solve $A\mathbf{x} = \mathbf{b}$ & $\mathbf{b}$ can be expressed as a lin combn of $(1, 2, 3)^T, (4, 5, 6)^T$.

$\therefore \mathbf{b} = (0, 1, 2)^T \in \text{Span}((1, 2, 3)^T, (4, 5, 6)^T)$.

Can find "Cartesian" form for subspaces.

e.g. 8 Find all $\mathbf{b}$ in $\text{Span}((1, 0, 2)^T, (3, 1, 2)^T)$.

A $\mathbf{b} = (b_1, b_2, b_3)^T$ is in the span iff we can solve

$x_1(1, 0, 2)^T + x_2(3, 1, 2)^T = (b_1, b_2, b_3)^T$.

Hence, $A\,\mathbf{x} = \mathbf{b}$ is solvable iff

Span here is a plane with Cart eqn

Span in other vector spaces

Reduce questions to solving a system of linear eqns as follows.

e.g. 9

$$\text{Is } \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \in \mathrm{Span}\left(\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 6 & 0 \end{pmatrix} \right)?$$

Equivalently, can you solve for $\lambda, \mu \in \mathbb{F}$

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + \mu \begin{pmatrix} 4 & 5 \\ 6 & 0 \end{pmatrix}.$$

Comparing entries we see we need to solve

Same eqns as in e.g.7 which is solvable. Thus it lies in the span.

e.g. 10 Which polynomials $b_1 + b_2x + b_3x^2$ lie in $\text{Span}(1 + 2x^2, 3 + x + 2x^2)$?

i.e. when can you solve

$b_1 + b_2x + b_3x^2 = \lambda(1 + 2x^2) + \mu(3 + x + 2x^2)$

Compare coefficients:

Same eqns as in e.g.8 so soln exists iff

Hence $b_1 + b_2x + b_3x^2$ is in span iff