Lecture 6: Vector Spaces

**Aim Lecture** Introduce vector spaces which provide natural context/language for describing linear phenomena.

**Motivation:** some linear phenomena

**In 3D-space**

Line has

param form

\[ x = a + \lambda v, \lambda \in \mathbb{R}. \]

**In \( \mathbb{R}^3 \):** 2 inequivalent non-zero linear equations has soln set of form

\[ x = a + \lambda v, \lambda \in \mathbb{R} \text{ for some choice of } a, v \in \mathbb{R}^3. \]

**Function space** Soln to \( \frac{dy}{dx} = 2x \)

is “line”

of quadratic
fns $y = x^2 + \lambda 1, \lambda \in \mathbb{R}$.

Wish to define notion of vector space where the expression $a + \lambda v$ makes sense so need notion of vector addn & scalar multn.

More gen, want standard rules of vector arithmetic to hold.

**Vector space axioms**

Let $F$ be a field like $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. A vector space over $F$ is

a. A set $V$ of elements called

& b. An addition law denoted $+$ which assigns to any $v, w \in V$ another vector $v + w \in V$,

This new vector is called

& c. A scalar multn law which assigns to any $v \in V$ and $\lambda \in F$ a vector $\lambda v \in V$, 
such that the following standard laws of vector arithmetic hold.

For any \( u, v, w \in V \),

1. Associative Law of Addition:
   \[(u + v) + w = (u + v) + w = \]
2. Commutative Law of Addition:
   \[u + v = u + v = \]
3. Existence of Zero: there’s a vector \( 0 \) called zero which satisfies the following special (defining) property: \( 0 + v = v \) (for all \( v \in V \)).
4. Existence of Negatives: there’s a vector \(-v\) called the negative of \( v \) which satisfies the defining property \( v + (-v) = \)
5. Associative Law for Scalar Multn:
   \[(\lambda \mu) v = \lambda (\mu v) . \]
6. \( 1v = v \).
7. Scalar Distributive:

\[(\lambda + \mu) \mathbf{v} = \lambda \mathbf{v} + \mu \mathbf{v}\]

8. Vector Distributive:

\[\lambda (\mathbf{v} + \mathbf{w}) =\]

**Subtle Pt:** Zero & negatives are uniquely defined by their defining property.

See notes §7.2, propn 1.

**Rem** The above laws are called the axioms for a vector space.

**Examples**

e.g. 1 \(\mathbb{R}^n\) is a vector space over \(\mathbb{R}\) when equipped with coordinate-wise addn & scalar multn i.e.

addition rule:

scalar multiplication rule:
e.g. 2 $\mathbb{C}^n = \{(z_1, \ldots, z_n) | z_1, \ldots, z_n \in \mathbb{C}\}$ is a vector space over $\mathbb{C}$ when equipped with coordinate-wise addition & scalar multiplication.

e.g. 3 Let $V$ = set of geometric vectors (i.e. “arrows”) in 3D-space $V$ is a vector space if we define
addition = usual head to tail addition of arrows
scalar multiplication = usual scaling “length” of vector.

e.g. 4 Let $F = \mathbb{R}$ or $\mathbb{C}$ or any other field.
Let $M_{mn}(F)$ be set of $m \times n$-matrices over $F$.
Define
vector addition to be matrix addition
scalar multiplication by scalar multiplication of matrices
i.e.

Then $M_{mn}(F)$ is a vector space over $\mathbb{R}$. 
You can check all axioms. Here we’ll only check existence of zero axiom:

Hence the zero matrix satisfies the defining property of a zero in a vector space i.e. it is the zero in $M_{mn}(\mathbb{F})$ so there exists a zero vector in $M_{mn}(\mathbb{F})$. This checks the axiom.

**N.B.** Also, negative of a matrix is the vect space negative.

Abstract vect space defns coincide with matrix defns.

**e.g.** 5 $X = $ non-empty set.

$\mathcal{R}[X] :=$ set of real-valued fns on $X$

$\mathcal{R}[X]$ is a vector space / $\mathbb{R}$ if define vector operations pointwise i.e.
vector addn: for $f, g \in \mathcal{R}[X]$

$(f + g)(x) = f(x) + g(x)$

scalar multn: for $\lambda \in \mathbb{R}, f \in \mathcal{R}[X]$

$(\lambda f)(x) = \lambda f(x)$

Then $\mathcal{R}[X]$ is a vector space / $\mathbb{R}$ with zero the constant fn with value 0.

**N.B.** Abstract vect space defns coincide with calculus defns.

Sim e.g. 6 $X = \text{non-empty set}$. 

$\mathcal{C}[X] := \text{set of } \mathbb{C}\text{-valued fns on } X$. 

$\mathcal{C}[X]$ is a vector space / $\mathbb{C}$ if define addn & scalar multn pointwise.

**Properties of vector spaces**

Let $V = \text{vector space / field } \mathbb{F}$. 

**Subtraction** Let $v, w \in V$. $-v$ is the only vector s.t.
We can define $w - v = w + (-v)$

Vect space axioms $\implies$ can algebraically manipulate vectors as you would geom vect.

**e.g. 8** For $v, w \in V$ simplify

$$2(3v + 4w) - 3w$$

$$= (2(3v) + 2(4w)) + (-3)w$$ \hspace{1cm} \text{vect distrib law}

$$= (6v + 8w) + (-3)w$$

$$= 6v + (8w + (-3)w)$$

$$= 6v + 5w.$$  

**ex** Simplify $2(v + w) + 3(v - w)$

**Propn** Let $V$ be a vector space / $\mathbb{F}$. For $\lambda \in \mathbb{F}, v \in V$

1. $\lambda \cdot 0 = 0$  
2. $0 \cdot v =$  
3. $(−1) \cdot v =$  
4. $\lambda \cdot v = 0 \implies$

Proof: 2) $0 \cdot v + 0 \cdot v = (0 + 0) \cdot v = 0 \cdot v$
Subtract $0 \mathbf{v}$ from both sides to see

$$0 \mathbf{v} + 0 \mathbf{v} - 0 \mathbf{v} = 0 \mathbf{v} - 0 \mathbf{v} = \mathbf{0}$$

So $0 \mathbf{v} = \mathbf{0}$.

1) is sim.

3) We check $(-1) \mathbf{v}$ satisfies defining property of negative i.e. $(-1) \mathbf{v} + \mathbf{v} = \mathbf{0}$.

$$(-1) \mathbf{v} + \mathbf{v} =$$

Hence $(-1) \mathbf{v} = -\mathbf{v}$.

4) If $\lambda \neq 0$ then

$$\mathbf{v} = 1 \mathbf{v} = (\lambda^{-1} \lambda) \mathbf{v} =$$

Hence either $\lambda = 0$ or $\mathbf{v} = \mathbf{0}$.

An exotic example

e.g. Twisted $\mathbb{C}^n$. $V = \mathbb{C}^n$ as a set.
Addn: usual coordinate-wise addn

New twisted scalar multn: for \( \lambda, z_1, \ldots, z_n \in \mathbb{C} \) define

\[
\lambda(z_1, \ldots, z_n) = (\bar{\lambda}z_1, \ldots, \bar{\lambda}z_n).
\]

\( V \) is a vector space over \( \mathbb{C} \).

Why? Axioms involving only addn checked as for usual \( \mathbb{C}^n \). Let’s check scalar distrib law: