Lecture 4: Complex Polynomials


**Remainder & Factor thm**

**Defn** A complex polynomial of degree $n$ is a fn $p : \mathbb{C} \rightarrow \mathbb{C}$ of the form

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

for some $a_0, \ldots, a_n \in \mathbb{C}$ with $a_n \neq 0$.

If the coeff $a_0, \ldots, a_n$ are real then we say $p(z)$ is a real polynomial.

Real poly will also refer to the real-valued fn $p : \mathbb{R} \rightarrow \mathbb{R}$ obtained by restricting the domain to $\mathbb{R}$.

**Remainder Thm** Let $p(z)$ be a poly & $\alpha \in \mathbb{C}$. The remainder $r$ on dividing $p(z)$ by $z - \alpha$ is $r = p(\alpha)$. 
Proof If quotient is \( q(z) \) then
\[
p(z) = \]
\[
\therefore p(\alpha) = \]

An immediate corollary is

**Factor Thm** Let \( p(z) \) be a poly \& \( \alpha \in \mathbb{C} \)

Then \( z - \alpha \) is a factor of \( p(z) \) iff \( p(\alpha) = 0 \)
i.e. \( \alpha \) is a root of \( p(z) \).

**Factorising over \( \mathbb{C} \)**

**Fund Thm of Algebra** (Gauss) Let \( p(z) = a_nz^n + \ldots + a_1z + a_0 \) be a complex poly of degree \( n > 0 \). Then \( p(z) \) has a complex root, so applying factor thm and induction we see we can express

\[
(*) \quad p(z) = a_n(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_n)
\]

where \( \alpha_1, \ldots, \alpha_n \) are all the roots of \( p(z) \) (sometimes repeated).

Furthermore, the factorisation in \((*)\) is unique up
to permuting factors.

**Defn** The number of times a root $\alpha_i$ occurs in the factorisation is called the multiplicity of the root.

**e.g. 1** $z^4 + 2z^2 + 1$

So $i, -i$ are roots of multiplicity 2.

**e.g. 2** Factorise $p(z) = 4 - z^6$ over $\mathbb{C}$.

**A** Use thm. Find solns to $p(z) = 0$ or equivalently $z^6 = 4$.

$|z|^6 = 6 \arg z = \Rightarrow \arg z =$

So roots are $z = \sqrt[3]{2}e^{\pm 2\pi i/3}, \sqrt[3]{2}e^{\pm \pi i/3}, \sqrt[3]{2}, \sqrt[3]{2}e^{i\pi} = -\sqrt[3]{2}$. 
\[ 4 - z^6 = -(z - \sqrt[3]{2}e^{2\pi i/3})(z - \sqrt[3]{2}e^{-2\pi i/3}) \]
\[ \times (z - \sqrt[3]{2}e^{\pi i/3})(z - \sqrt[3]{2}e^{-\pi i/3})(z - \sqrt{2})(z + \sqrt{2}) \]

**Factorising over \( \mathbb{R} \)**

Use

**Propn**

a) Let \( p(z) = \sum a_j z^j \) be a real poly & \( z = \alpha \) be a complex root. Then \( \bar{\alpha} \) is also a root.

b) \((z - \alpha)(z - \bar{\alpha}) = z^2 - (2\text{Re } \alpha)z + |\alpha|^2\)

which is real.

Proof 1): If \( 0 = p(\alpha) = \)
then \( 0 = \sum \)

**e.g. 2 cont’d** Factorise \( 4 - z^6 \) over \( \mathbb{R} \).

**A** Collect factors corresp to complex conjugate roots.

\((z - \sqrt[3]{2}e^{\pi i/3})(z - \sqrt[3]{2}e^{-\pi i/3}) = \)
\( z^2 - (2\text{Re } \sqrt[3]{2}e^{\pi i/3})z + |\sqrt[3]{2}e^{\pi i/3}|^2 \)
\[ z^2 - \sqrt[3]{2}z + \sqrt[3]{4}. \]

Sim \( (z - \sqrt[3]{2}e^{2\pi i/3})(z - \sqrt[3]{2}e^{-2\pi i/3}) = \]

\[ 4 - z^6 = -(z - \sqrt[3]{2})(z + \sqrt[3]{2}) \times (z^2 - \sqrt[3]{2}z + \sqrt[3]{4})(z^2 + \sqrt[3]{2}z + \sqrt[3]{4}). \]

**Rem** This procedure shows you can factorise any real poly into real linear & quadratic factors.

**Application to polynomial interpolation**

**Corollary** a) If poly \( p(z), q(z) \) have degrees \( \leq n \) & agree on \( n + 1 \) different values \( z = \alpha_1, \ldots, \alpha_{n+1} \) i.e. \( p(\alpha_1) = \)
then \( p(z) = q(z) \).

b) Any 2 poly which agree on an infinite set are the same.

Proof: Clear a) \( \implies \) b). Note \( g(z) := p(z) - q(z) \)
has degree $\leq n$.

It also has more than $n$ roots (namely, $\alpha_1, \ldots, \alpha_{n+1}$) so it must be 0.

**e.g. 2** Given 3 distinct points $(x_1, y_1), (x_2, y_2)$ & $(x_3, y_3)$, there is at most 1 parabola of the form

$$y = p(x)$$

going through those points.

Why? If $y = q(x)$ also went through those points then

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**Symmetric polynomials in the roots**

**Defn** A poly $p(x_1, \ldots, x_n)$ in var $x_1, \ldots, x_n$ is symmetric if it remains the same on swapping any 2 variables.

**e.g. 4** In 3 var,
\[ x_1 + x_2 + x_3 \] is symmetric
\[ x_1x_2 + x_2x_3 \] is not because

**e.g. 5** Suppose \( z^2 + bz + c \) has roots \( \alpha, \beta \in \mathbb{C} \).
\[
z^2 + bz + c = (z - \alpha)(z - \beta) = z^2 - (\alpha + \beta)z + \alpha\beta.
\]
\[ \implies \text{sum of roots} = \]
& \text{product of roots} =

More generally,

**Prop** Let \( \alpha_1, \ldots, \alpha_n \) be the roots (with multiplicity) of
\[
p(z) = a_0 + a_1z + \ldots + a_nz^n.
\]
Then \( \frac{a_{n-j}}{a_n} = (-1)^j \) sum of all products of \( j \) roots.

**Proof:** Just expand

**N.B.** The \( a_i \)'s are symmetric poly in the \( \alpha_i \)'s since
\( p(z) \) remains the same on swapping any two linear
factors in its factorisation.

**Thm** Any symmetric poly in the roots of $p(z)$ is a poly in the coeff of $p(z)$.

No proof.

**e.g.** 6 If $z^3 + 2z^2 + 3z + 4$ has roots $\alpha_1, \alpha_2, \alpha_3$ then

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2$$

is symmetric and equals