Lecture 20: DEs and diagonalisation

**Aim lecture** See how diagonalisation is useful for solving differential eqns.

**Motivation**

**e.g. 1** \( y_1(t) = \) popn of hobbits in \( y_2(t) = \) popn of orcs

If 2 popn kept separate as here then popn growth governed by a pair of DEs which typically looks something like:

\[
\begin{align*}
    y_1'(t) &= 3y_1(t) \\
    y_2'(t) &= 2y_2(t)
\end{align*}
\]  

(*)

Soln: Easy, solve 2 eqns separately

\( y_1(t) = \)

Suppose now we put the two popns together in
Typical DEs describing popn growth is
\[ y_1'(t) = 3y_1(t) - 2y_2(t) \]
\[ y_2'(t) = -y_1(t) + 2y_2(t) \] (†)

These are “coupled” DEs i.e. \( y_1', \ y_2' \) each depend on both \( y_1 & y_2 \). We’ll use diag to

**N.B.** Growth rate of hobbit popn depends positively on hobbit popn &

**Notn** \( y(t) = \)

\[ y'(t) = \frac{dy}{dt} := \]

In e.g. 1, we can write
\[
\begin{pmatrix}
3y_1 - 2y_2 \\
-y_1 + 2y_2
\end{pmatrix}
= \\
\begin{pmatrix}
3 -2 \\
-1 2
\end{pmatrix}.
\]

so \( y'(t) = A y(t) \) where \( A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \).

Note In decoupled case (*) above, still have \( y'(t) = A y(t) \) but now
\[ A = \]

Diagonalisation & decoupling DEs

Consider more generally
\[ y(t) = \]

& system of \( n \) linear DEs
\[ y'(t) = A y(t) \]
where \( A \in M_{n,n}(\mathbb{R}) \).
Lemma For $C \in M_{n,n}(\mathbb{R})$
\[
\frac{d}{dt}(C\,y) = C\frac{dy}{dt}.
\]

Proof: Clear from case $n = 2$. Suppose
\[
C = \begin{pmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{pmatrix}
\]
\[
\frac{d}{dt}(C\,y) =
\]

To solve $y'(t) = A\,y(t)$, suppose we can diag $A = M\,D\,M^{-1}$ with $M = (f_1 \ldots f_n)$
\& $D =$
**Thm** 1) If we change variables to
\[ x(t) = M^{-1} y(t) \]
then get decoupled eqn
\[(*) \quad \frac{dx}{dt} = D x \]

2) Soln to (*) is
\[ x_i(t) = \alpha_i e^{\lambda_i t} \text{ for } i = 1, \ldots, n \]
& scalars \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \).

3) Soln to original DE \( y'(t) = Ay(t) \) is
\[ y(t) = M x(t) = \alpha_1 e^{\lambda_1 t} f_1 + \ldots + \alpha_n e^{\lambda_n t} f_n \]
where \( f_1, \ldots, f_n \) are e-vectors with correspond e-values \( \lambda_1, \ldots, \lambda_n \).

Proof: 1) \( y'(t) = Ay = MDM^{-1} y = MD x \).
Also, lemma \( \implies \frac{d}{dt}(M x(t)) = \)

Equating & noting \( M \) invertible we see \( \frac{dx}{dt} = D x \).

2) (*) corresponds to system of linear DEs
cont’d

3) Just multiply matrices.
\[ y(t) = M \, x(t) = (f_1 \ldots f_n) \]

\[ = \alpha_1 e^{\lambda_1 t} f_1 + \ldots + \alpha_n e^{\lambda_n t} f_n \]

Example

e.g. 1 completed

We diag \( A \)

\[ 0 = \det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -2 \\ -1 & 2 - \lambda \end{vmatrix} \]

The e-values are 4,1.

E-vectors?
\[ \lambda = 4 : \ker(A - \lambda I) = \]

An e-vector is
\[ \lambda = 1 : \ker(A - \lambda I) = \]

An e-vector is
Thm 3) \[ \implies \]
\[ y(t) = \]

for some scalars \( \alpha_1, \alpha_2 \).
i.e. \[ y_1(t) = \]
\[ y_2(t) = \]

**e.g. 2** Suppose in e.g. 1 that initial popn is \[ y(0)^T = (4000, 1000) \]. Solve the IVP.

Ans: We need only solve for \( \alpha_1, \alpha_2 \).
From Gaussian elim or guessing see

\[ \alpha_1 = \]

The soln is thus

\[ y(t) = \]

**e.g. 3** What happens in e.g. 2 as \( t \to \infty \)?

Nasty hobbits!

**N.B.** Key to limiting behaviour is e-value of max magnitude.

**Second order DEs**

We can convert any 2nd order const coeff linear ODE into a pair of linear ODEs in 2 var as in following
E.g. 4 Solve IVP

\[ y'' - 3y' + 2y = 0 \ , \quad y(0) = 2, \ y'(0) = 3 \]

Ans: Let \( y_1 = y, y_2 = y' \)

\[ y_1' = y' = y_2 \]
\[ y_2' = y'' = -2y + 3y' = -2y_1 + 3y_2 \]

i.e. \( y' = \)

Diag \( A = \)

\[ \det(A - \lambda I) = \]

Hence e-values are 2,1.

E-vectors:

\( \lambda = 2 : \ker(A - \lambda I) = \)
An e-vector is
\[ \lambda = 1 : \ker(A - \lambda I) = \]

An e-vector is
Hence, (from thm 3)) general soln is
\[ y(t) = \]

Need now find integration constants.