
**Aim Lecture** Find simple methods to compute powers & extract roots using complex exponential fn.

**Euler’s formula for complex exp fn**

**Lemma** \((\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi)\)

\[= \cos(\theta + \phi) + i \sin(\theta + \phi).\]

**Proof** LHS = \((\cos \theta \cos \phi - \sin \theta \sin \phi)\)

\[+ i(\sin \theta \cos \phi + \cos \theta \sin \phi)\]

**Euler’s Formula** For \(\theta \in \mathbb{R}\)

\(e^{i\theta} := \cos \theta + i \sin \theta.\)

More gen for \(a, b \in \mathbb{R}\)

\(e^{a+bi} :=\)

N.B. Comparing with polar form we see
\[ |e^{a+bi}| = \]
\[ \text{Arg } e^{a+bi} \]

**e.g.** \[ e^{i\pi/2} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = \]
Can also see this from Argand diagram

---

**Properties of exponential fn**

**Q** Why is Euler’s defn sensible?

**One** **A** Have desirable

**Facts**

1. It recovers real exp fn when \( z \in \mathbb{R} \).
2. \( e^{z+z'} = e^z e^{z'} \)
3. \( (e^z)^{-1} = e^{-z} \)
4. \( \frac{e^z}{e^{z'}} = e^{z-z'} \)
5. For \( n \in \mathbb{Z} \), \( (e^z)^n = e^{nz} \)

**Proof:**

1. \( e^{a+0i} = e^a (\cos 0 + i \sin 0) = \)
2. Write \( z = a + bi, \ z' = a' + b'i \)

LHS = \( e^{a+bi+a'+b'i} = e^{(a+a')+(b+b')} \)

= \( e^{a+a'}(\cos(b + b') + i \sin(b + b')) \)

\text{lemma} \quad \frac{e^a e^{a'}(\cos b + i \sin b)(\cos b' + i \sin b')}{e^z e^{z'}}.

3. \( e^{-z} e^z = e^0 = 1 \) so


5. For \( n \geq 1 \) it follows by

Clear for \( n = 0 \).

For \( n < 0 \),

\( (e^z)^n = \frac{1}{(e^z)^{-n}} = \frac{1}{e^{-nz}} = e^{nz} \).

An immediate corollary is

**De Moivre’s Thm**

\( (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \)

Proof: LHS = \( (e^{i\theta})^n = \)

**Fact** 1. For \( n \in \mathbb{Z} \),
\[ e^{i(\theta+2n\pi)} = e^{i\theta} \]

2. \[ e^{i\theta} = e^{i\theta'} \implies \theta - \theta' = 2n\pi \] for some \( n \in \mathbb{Z} \).

Proof: 1. holds as \( \cos, \sin \) have period \( 2\pi \).

2. \[ e^{i\theta} = e^{i\theta'} \implies e^{i(\theta-\theta')} = 1 \]
\[ \implies \cos(\theta - \theta') = 1 \] so \( \theta - \theta' \) is a multiple of \( 2\pi \).

**e.g.** \( e^{-i\pi/2} = e^{i3\pi/2} \) as can also been seen from

**Products in polar form**

For \( r, \theta \in \mathbb{R}, \)

\[ r(\cos \theta + i \sin \theta) = re^{i\theta} \]

This is alternate polar form.

N.B. \( re^{i\theta} = r'e^{i\theta'} \) iff \( r = r' \) \& \( \frac{\theta - \theta'}{2\pi} \in \mathbb{Z} \).

**Consequences** For \( z, w \in \mathbb{C} \)

1. \[ \frac{z}{w} = \frac{|z|}{|w|}, \quad |zw| = |z||w| \].
2. \( \text{Arg } zw = \text{Arg } z + \text{Arg } w + 2n\pi \)

for some \( n \in \mathbb{Z} \).

\( \text{Arg } \frac{z}{w} = \)

3. \( |z^n| = \)

4. \( \text{Arg } z^n = \)

**Proof** half of 1 & 2 only. Others sim.

Let \( z = re^{i\theta}, w = se^{i\phi} \)

\( \frac{z}{w} = \)

This is the polar form for \( \frac{z}{w} \)

Hence, \( |\frac{z}{w}| = \)

\( \text{Arg } \frac{z}{w} \)
e.g. Let $z = -1 + i$ so $\text{Arg } z =$

Then $z^2 =$

so $\text{Arg } z^2 =$

Note 2 $\text{Arg } z = \text{Arg } z^2 + 2\pi$.

e.g. Let $z = -1 + i, w = 3e^{-2i}$. We find the polar form of $zw$.

$|zw| =$

\text{Arg } zw

Powers via polar form

e.g. 1 Find $(\sqrt{3} - i)^{100}$

Dumb method

Good Method: Write $z = \sqrt{3} - i$ in polar form

$|z| =$
Arg $z = \arg z$.

\[ \therefore z = 2e^{i\pi/6} \]

\[ z^{100} = 2^{100}e^{i100\pi/6} = 2^{100}e^{i50\pi/3} \]

\[ = 2^{100}e^{2\pi i/3} \]

as $e^{48\pi i/3} = e^{16\pi i} = \]

\[ = 2^{100}(\]

\[ = -2^{99} \]

$n$-th roots via polar form

e.g. 2 Find all cube roots of $8i$.

A Suppose $z^3 = 8i$, $z = re^{i\theta}$

Polar form of $8i =$

\[ z^3 = r^3e^{i3\theta} \]

Equate moduli:

Equate arg: $3\theta = \frac{\pi}{2} + 2n\pi$

for some $n \in \mathbb{Z}$.
N.B. $-\pi < \theta \leq \pi \implies -3\pi < 3\theta \leq 3\pi$
Hence, $3\theta = $
\[ \therefore \theta = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6} \]
So the three cube roots are
\[ z = re^{i\theta} \]

Let’s plot the 3 cube roots

Roots of unity

Method in e.g. 2 gives

**Thm 1** The $n$-th roots of unity i.e. solns to $z^n = 1$ are

\[ z = 1, e^{2\pi i/n}, e^{4\pi i/n}, \ldots, e^{2\pi i(n-1)/n} \]
Check: For $k \in \mathbb{Z}$, $z = e^{i2k\pi/n} \implies z^n = z$

**Thm 2** Let $\omega := e^{2\pi i/n}$,

If $z_0$ is an $n$-th root of $\alpha \in \mathbb{C}$ then the $n$-th roots of $\alpha$ are

$z = z_0$,

Proof: $z_0\omega^j$ is a root $\because$

$(z_0\omega^j)^n = (z_0)^n\omega^{jn} = (z_0)^n$

Conversely, if $z_1^n = \alpha$

$(\frac{z_1}{z_0})^n$

Thm 1 $\implies$

so $z_1 = $
Multn & rotation

Shows multn by $e^{i\phi}$ rotates anti-clockwise by $\phi$.

Gives geom interpretation of thm 2. If $z_0$ is an $n$-th root of $\alpha$ then the other roots are obtained by rotating $\frac{2\pi}{n}, \frac{4\pi}{n}, \ldots$.

$n$-th roots are equally spaced around a circle.

Square roots via cartesian form
**e.g. 3** Solve $z^2 = -5 + 12i$

**A** Let $z = a + bi, a, b \in \mathbb{R}$

$z^2 = (a^2 - b^2) + 2abi$

Equate real & imag parts

Solve simultaneously by guessing or elim $a$ using

$a^2 + b^2 = |z|^2 = |z^2| = |-5 + 12i| = \sqrt{5^2 + 12^2} = 13$

$\therefore 2a^2 =$

$b =$

So square roots are $z = a + bi = \pm(2 + 3i)$.

**Quadratic formula**

$az^2 + bz + c = 0, \quad a, b, c \in \mathbb{C}$

has solns
e.g. 4 Solve $z^2 + (-4 + i)z + (5 - 5i) = 0$.

discriminant = $b^2 - 4ac = (-4 + i)^2 - 4(5 - 5i)$

= $16 - 8i + i^2 - 20 + 20i = -5 + 12i$

From e.g. 3 we see

$z = \frac{1}{2}(4 - i \pm (2 + 3i))$

= $3 + i$ or $1 - 2i$