Lecture 15: Geometry & Algebra of linear maps.

Aim Lecture Study some geom examples of lin maps. See how the algebra of linear maps mimics the algebra of matrices.

Visualising some lin maps

e.g. 1 Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the lin map assoc to matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & .5 \end{pmatrix}$ i.e. $T(x, y)^T = (2x, .5y)^T$.

We can plot some values of $T$, $T e_1 = $, $T e_2 =$

Looking at a unit square:

\[ \therefore T \text{ stretches out horizontally by a factor of } 2 \]
Orthogonal projection

Let $\mathbf{u} \in \mathbb{R}^n$ be unit vector.

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be projn onto line $L := \text{Span}(\mathbf{u})$.

Recall $T \mathbf{v} = \text{proj}_\mathbf{u} \mathbf{v} = (\mathbf{v} \cdot \mathbf{u}) \mathbf{u}$.

$T$ is linear.

Proof 1: Add Condn: for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$

$T(\mathbf{v} + \mathbf{w}) = ((\mathbf{v} + \mathbf{w}) \cdot \mathbf{u}) \mathbf{u}$

$= (\mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}) \mathbf{u}$

$= (\mathbf{v} \cdot \mathbf{u}) \mathbf{u} + (\mathbf{w} \cdot \mathbf{u}) \mathbf{u}$

$= T \mathbf{v} + T \mathbf{w}$
so addn condn holds.

Scalar Multn: For \( \lambda \in \mathbb{R} \),
\[
T(\lambda \mathbf{v}) =
\]

Hence scalar multn also holds & \( T \) is linear.

Proof 2: Geometric. e.g. see picture for addn condn.

Proof 3: \( T \mathbf{v} = (\mathbf{u}^T \mathbf{v}) \mathbf{u} = \mathbf{u} \mathbf{u}^T \mathbf{v} \)
(since \( \mathbf{u}^T \mathbf{v} \) is a scalar so commutes with \( \mathbf{u} \)).
\[
\therefore T \text{ is the lin map assoc to the matrix } A =
\]

Linear combns of lin maps

**Defn 1** Let \( T, T' : V \rightarrow W \) be lin maps.

Let \( \lambda, \mu \in \mathbb{F} \). Define new maps

Sum: \( T + T' : V \rightarrow W \) by
\[
(T + T') \mathbf{v} = T \mathbf{v} + T' \mathbf{v}.
\]
scalar multiple: \( \lambda T : V \rightarrow W \) by
\[
(\lambda T) v = \lambda(T v).
\]
Linear Combn: \( \lambda T + \mu T' : V \rightarrow W \) by
\[
(\lambda T + \mu T') v = \lambda(T v) + \mu(T' v).
\]
**Propn 1** \( T + T', \lambda T \) are linear too. Hence so is \( \lambda T + \mu T' \).

Proof: Check \( \lambda T \) lin: for \( v, w \in V, \alpha \in \mathbb{F} \)
Add Condn: \( (\lambda T)(v + w) \) def
\[
= \lambda(T v + w) = \lambda(T v)
\]
so addn condn holds.
Scalar Multn Condn: \( (\lambda T)(\alpha v) \)
\[
= \lambda(T(\alpha v)) = \lambda(\alpha T v)
= \alpha \lambda(T v) = \alpha((\lambda T) v)
\]
so scalar multn condn also holds & \( T \) is linear.
Check $T + T'$ linear: Can prove as above. Here we only prove in case $V = \mathbb{F}^n, W = \mathbb{F}^m$ so by matrix reprn thm there are $A, B \in M_{mn}(\mathbb{F})$ such that for any $v \in V$ we have

$$T v = A v, \quad T' v = B v.$$ 

$$(T + T') v = T v + T' v = A v + B v = (A + B) v.$$ 

$T + T'$ is lin map assoc to the matrix $A + B$ so is linear.

Proof gives

**Formula 1** $T_A + T_B = T_{A+B}$

Sim $\lambda T_A + \mu T_B = T_{\lambda A + \mu B}$.

where $T_A, T_B$ etc are lin maps assoc with matrices $A, B$ etc.

Application to orthogonal projections

e.g. 2 Identity linear map
For vect space $V$, let $\text{id} : V \longrightarrow V$ be the function $\text{id} \, \mathbf{v} = \mathbf{v}$. It is linear (CHECK!).

$\text{id} : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ is represented by

since $\text{id} \, \mathbf{v} = \mathbf{v} =

\textbf{e.g. 3} \text{ Let } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3 \text{ be an o/n basis. }

Let $P_i := \text{projn onto line } \text{Span}(\mathbf{v}_i)$.

What’s $P_{12} := \text{projn onto plane } \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$?

Method 1: $P_{12} \, \mathbf{v} = \mathbf{v} - \text{proj}_{\mathbf{v}_3} \mathbf{v}$

$\text{id} \, \mathbf{v} - P_3 \, \mathbf{v} = (\text{id} - P_3) \, \mathbf{v}$.

$\therefore P_{12} = \text{id} - P_3$

so is linear being a lin combn of the lin maps $\text{id} \, &$
Method 2: also $P_{12} \mathbf{v} = P_1 \mathbf{v} + P_2 \mathbf{v}$

Above show $P_{12}$ is matrix multn by

$I_3 - \mathbf{v}_3 \mathbf{v}_3^T =$

Composing & inverting linear maps

**Propn 2** Let $T : V \rightarrow W, S : U \rightarrow V$ be linear maps. Then $T \circ S : U \rightarrow W$ is linear.

(Recall $(T \circ S) \mathbf{u} = T(S \mathbf{u})$.)

Proof: ex in checking axioms, though following formula should convince you it’s true.

**Formula 2** For $A \in M_{lm}(\mathbb{F}), B \in M_{mn}(\mathbb{F})$ we have $T_A \circ T_B = T_{AB}$.

Why? For any $\mathbf{u} \in \mathbb{F}^n$ we have

$(T_A \circ T_B) \mathbf{u} = T_A(T_B \mathbf{u}) = T_A(B \mathbf{u}) = AB \mathbf{u} =$
Propn 3 If $T : V \longrightarrow W$ is linear and invertible then

its inverse $T^{-1} : W \longrightarrow V$ is also linear.

Proof: ex in checking axioms though following formula should convince you it’s true.

Formula 3 For an invertible matrix $A$,

\[(T_A)^{-1} = T_{A^{-1}}.\]

Why? Given vectors $x, y$ with $T_A x = y$, we see $A x = y$.

\[\therefore x = A^{-1} y = T_{A^{-1}} y.\]

This shows $(T_A)^{-1} = T_{A^{-1}}$.

Example of rotations

Let $T_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be anti-clockwise rotation
about $(0, 0)$ through angle $\theta$. 

What’s $T\left(\begin{array}{c} x \\ y \end{array}\right)$?

On can check geom that $T_\theta$ is linear e.g. scalar multn condn $T(\lambda \mathbf{v}) = \lambda T \mathbf{v}$ for $\lambda \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^2$ follows from picture.

Matrix reprn thm $\implies$ $T_\theta$ must be the lin map assoc to the matrix

$R_\theta := (T \mathbf{e}_1 T \mathbf{e}_2)$.

From picture
see \( T \mathbf{e}_1 = \)

so \( T \) is the lin map assoc to the matrix

\[
R_{\theta} = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]

**e.g.** 4 Prop & formula 3 can be easily verified directly for the rotation \( T_{\theta} \).

Geometrically we see \( T_{\theta} \) is invertible with “inverse” rotation \( T_{\theta}^{-1} = \)

Hence the inverse is linear too (being another rotn).

We now check the matrix representing \( T_{\theta}^{-1} \) is indeed \( R_{\theta}^{-1} \).

\[
R_{\theta}^{-1} = \left( \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \right)^{-1}
= \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]
\[
\begin{pmatrix}
\cos(-\theta) & \sin(-\theta) \\
\sin(-\theta) & \cos(-\theta)
\end{pmatrix}
\]
which is indeed the matrix representing...