Lecture 13: Data fitting. Review functions.

**Aim Lecture** We observe role of linear algebra in data fitting. We review invertible fns.

**Motivational example**

**e.g. 1** Lex believes

Experiment:

<table>
<thead>
<tr>
<th>$r$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strength</td>
<td>9.7</td>
<td>9.2</td>
<td>7.9</td>
<td>4.9</td>
</tr>
</tbody>
</table>

Typical questions:

Interpolation: estimate strength when $r$ is between data values, e.g. $r = 3.5$

Extrapolation: estimate strength when $r$ is outside range of data, e.g. $r = 5$.

**Objective** Find a cubic fn

$$y(r) = a_0 + a_1r + a_2r^2 + a_3r^3$$ such that
\( y(1) = 9.7, \ y(2) = \)

**Point** These 4 eqns give 4 lin eqns in \( a_0, a_1, a_2, a_3 \) with which to determine poly \( y(r) \). Can then use \( y(r) \) to estimate strength for other values of \( r \) other than 1, 2, 3, 4.

**Polynomial interpolation**

Consider data points \((t_0, y_0), \ldots, (t_n, y_n) \in \mathbb{R}^2\) where \( t_0, \ldots, t_n \) are distinct.

Seek polynomial

\[
y(t) = \lambda_0 + \lambda_1 t + \ldots + \lambda_n t^n
\]

s.t. \( y_i = y(t_i) \)

**Notn:** \( t = (t_0, t_1, \ldots, t_n)^T \)

\[
y(t) = (y(t_0), y(t_1), \ldots, y(t_n))^T
\]

\( y = \)
\( \lambda = \)

Want to find fn \( y(t) \) with \( y(t) = y \). If 

\[
A := \begin{pmatrix}
1 & t_0 & t_0^2 & \ldots & t_0^n \\
1 & t_1 & t_1^2 & \ldots & t_1^n \\
\vdots \\
1 & t_n & t_n^2 & \ldots & t_n^n
\end{pmatrix}
\]

then \( A \lambda = \)

\( = y(t) \)

**Upshot** The polynomials \( y(t) \) which “fit the data”, i.e. with \( y(t) = y \) 

are those whose coeff \( \lambda_0, \lambda_1, \ldots, \lambda_n \) solve \( A \lambda = y \).

**Thm 1** \( A \) is invertible so there’s a unique poly 
\( y(t) = \lambda_0 + \lambda_1 t + \ldots + \lambda_n t^n \)

in \( \mathbb{P}_n \) s.t. \( y(t_i) = y_i \). It’s coeff are given \( \lambda = \)

Proof: Cor a) lect 4 \( \implies \) at most 1 poly of degree
\( \leq n \) can fit the data.

\[ \therefore \text{if } A \lambda = \mathbf{y} \text{ has solns then it is unique.} \]

\[ \therefore \text{columns of} \]

\[ \therefore \text{row-echelon form has all} \]

But \( A \) is square so all rows are leading too and \( A \) must be invertible.

\textbf{N.B.} \( \dim \mathbb{P}_n = \text{no. data points.} \)

\textbf{e.g. 1 cont’d} see MATLAB lexl.m

Data points: \( (t_i, y_i) = \)
\( (1, 9.7), (2, 9.2), (3, 7.9), (4, 4.9) \)

\( \mathbf{t} = (1, 2, 3, 4)^T, \mathbf{y} = \)

\( A = \)

The coeff of the cubic fn \( y(t) \) which fits the data is

\( \lambda = A^{-1} \mathbf{y} = \)
i.e. \( y(t) = \) 

\textbf{Lagrange polynomials}

Previously found \( y(t) \) by finding
its coeff = coords wrt basis

Here construct more natural basis.

Consider \( \mathbf{t} = (t_0, \ldots, t_n)^T \) with \( t_i \) distinct.

For \( j = 0, \ldots, n \) consider Lagrange polynomials
\[
P_j(t) := \prod_{k \neq j} \frac{t - t_k}{t_j - t_k} \in \mathbb{P}_n
\]

This is a product of \( n \) linear factors.

\textbf{e.g. 2} If \( \mathbf{t} = (1, 2, 3) \) then

\[
P_0(t) = \frac{(t-t_1) \ (t-t_2)}{(t_0-t_1) \ (t_0-t_2)}
\]

\[
P_1(t) = \]

\[
P_2(t) = \]

\textbf{Thm 2 a)} \( P_j(t_j) = \)
b) \( P_j(t_i) = \)

c) \( B = \{ P_0(t), \ldots, P_n(t) \} \) is a basis for \( \mathbb{P}_n \).

d) If \( y(t) \in \mathbb{P}_n \) fits the data i.e. \( y(t) = y \) then

\[
(*) \quad y(t) = y_0 P_0(t) + \ldots + y_n P_n(t).
\]

i.e. \( [y(t)]_B = (y_0, \ldots, y_n)^T = y \).

Proof: a) & b) follow on substn.

c) \( \dim \mathbb{P}_n = n + 1 \). \( B \subset \mathbb{P}_n \) also has \( n + 1 \) vectors so suffice show it is lin indep.

Suppose \( \lambda_0 P_0(t) + \ldots + \lambda_n P_n(t) = 0 \).

For any \( i = 0, \ldots, n \),

\[
0 = \lambda_0 P_0(t_i) + \ldots + \lambda_i P_i(t_i) + \ldots + \lambda_n P_n(t_i)
\]

\[
\therefore P_0(t), \ldots, P_n(t) \text{ are lin indep. } \therefore B \text{ is a basis.}
\]

d) Just note

\[
y_0 P_0(t_i) +
\]

so both sides of (*) have the same values for the
$n+1$ inputs $t_0, \ldots, t_n$. But both sides are also polys of degree $\leq n$ so, since they agree for $n+1$ different values, they must be the same.

**N.B.** Thm 2d) means don’t need to solve eqns to find $y(t)$. BUT you need to work to compute $P_0(t), \ldots, P_n(t)$.

**e.g. 1 again** If $t = (1, 2, 3, 4)^T$ then desired cubic fn can also be written as

$$y(t) = 9.7P$$

$P_0(t), P_1(t), P_2(t), P_3(t)$ are the appropriate Lagrange polys.

**Interpolation by general functions**

Gen setup. Consider data points

$$(t_1, y_1), \ldots, (t_n, y_n)$$

Let $\phi_1, \ldots, \phi_n \in \mathcal{R}[\mathbb{R}]$ be lin indep.

**Q** Find $y(t) \in \text{Span}(\phi_1, \ldots, \phi_n)$
s.t. $y(t_i) = y_i$ for all $i$ i.e. $y(t) = y$.

As in poly case, write

$y(t) = \lambda_1 \phi_1 + \ldots + \lambda_n \phi_n.$

Then $y(t) = A\lambda$ so solving for $y(t)$ amounts to solving $A\lambda = y$.

**e.g. 2** Market with price trend anticipation.

Price $p(t)$ at time $t$ governed by a 2nd order ODE like

$$\frac{d^2 p}{dt^2} - 3\frac{dp}{dt} + 2p = 0$$

Find $p(t)$ if $p(0) = 7, p(1) = 8$

Char eqn $\lambda^2 - 3\lambda + 2 = 0$

$$\implies p(t) = \lambda_1 e^t + \lambda_2 e^{2t} \in \text{Span}(e^t, e^{2t})$$

Now just solve for $\lambda_1, \lambda_2$ using
Invertible functions

Recall following defns from calculus regarding a function $f : X \to Y$.

**Defn** 1) We say that $f$ is one-to-one (1-1) or injective if for any $y \in Y$, the soln to $f(x) = y$ is
i.e. $f(x) = f(x') \implies$

2) $f : X \to Y$ is onto or surjective if for any $y \in Y$, a soln to $f(x) = y$

i.e. $\text{im } f$

Recall also from calculus

**Facts** a) A fn $f : X \to Y$ is invertible iff $f$ is 1-1 &

b) In this case, the eqn $f(x) = y$ always has a
denoted $x = c) f \circ f^{-1} =

e.g. \ 3 \ f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \ \text{defined by} \ f(x_1, x_2) = (x_2, 2x_1) \ \text{is invertible with inverse} \ f(y_1, y_2) =$