Lecture 11: Dimension

Aim Lecture A basis \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \) of \( V \) gives \( n \)-dim coord system. Suggests \( V \) is \( n \)-dimensional. Need theory to ensure any two bases have the same number of vectors.

How does \( \text{Span}(S) \) vary with \( S \)

Prop \( V = \) vector space/ field \( \mathbb{F} \)
Let \( S_1 \subseteq S_2 \) be finite subsets of \( V \). Then \( \text{Span}(S_1) \subseteq \text{Span}(S_2) \).
Proof: We may suppose \( S_1 = \{ \mathbf{v}_1, \ldots, \mathbf{v}_m \}, S_2 = \{ \mathbf{v}_1, \ldots, \mathbf{v}_m, \ldots, \mathbf{v}_n \} \).

Need show every lin combn of \( S_1 \) is also a lin combn of \( S_2 \). Consider a lin combn of \( S_1 \)
\[
\lambda_1 \mathbf{v}_1 + \ldots + \lambda_m \mathbf{v}_m = \\
\lambda_1 \mathbf{v}_1 + \ldots + \lambda_m \mathbf{v}_m + 0 \mathbf{v}_{m+1} + \ldots + 0 \mathbf{v}_n
\]
which is also a lin combn of $S_2$. Hence every vector in Span($S_1$) is also in Span($S_2$) and the prop is proved.

Thm 1 Let $v, v_1, \ldots, v_n \in V = \text{vector space/field } F$. Then

\[ (*) \quad \text{Span}(v_1, \ldots, v_n, v) = \text{Span}(v_1, \ldots, v_n) \]

iff $v \in \text{Span}(v_1, \ldots, v_n)$.

Proof: ( $\implies$ ) If $(*)$ holds then

$v \in \text{Span}(v_1, \ldots, v_n, v) = \text{Span}(v_1, \ldots, v_n)$.

( $\impliedby$ ) Suppose $v \in \text{Span}(v_1, \ldots, v_n)$ so say

$v = \alpha_1 v_1 + \ldots + \alpha_n v_n$

for some scalars $\alpha_1, \ldots, \alpha_n$. We first show

$\text{Span}(v_1, \ldots, v_n, v) \subseteq \text{Span}(v_1, \ldots, v_n)$. An arbitrary elt of

$\text{Span}(v_1, \ldots, v_n, v)$ has form

$\lambda_1 v_1 + \ldots + \lambda_n v_n + \lambda v =$
= (\lambda_1 + \alpha_1 \lambda) \mathbf{v}_1 + \ldots + (\lambda_n + \alpha_n \lambda) \mathbf{v}_n
which lies in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n).
Hence, \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}) \subseteq \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)
Prop 4 \implies
\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}) \supseteq \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)
The two inclusions show \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}) = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n).

Shrinking spanning sets to bases

\textbf{Rem} If S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} lin depend then some
\mathbf{v}_i \in \text{Span}(S - \{\mathbf{v}_i\}) (by alt charn of lin depend).
Prop \implies
\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) =
\text{Span}(\mathbf{v}_1, \ldots \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n)
i.e. can omit \mathbf{v}_i without from S without shrinking
its span.

\textbf{e.g.} 1 Suppose S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} all lie in plane
$P \subseteq \mathbb{R}^3$ & no two are parallel.

Span$(S) = P$. Above remark shows that you can remove any of the 3 vectors from $S$ and the remaining two will still span $P$. This is clear geometrically from picture.

**Thm 2** Suppose $S \subset V$ is a finite spanning set for $V$. Then some subset of $S$ is a basis for $S$.

Why? We run following algorithm which keeps deleting vectors from $S$ until you arrive at a basis.

Step 1: Ask if some $v \in S$ is a lin combn of other vectors in $S$.

Step 2: If no, then $S$ is lin indep by alt charn of lin
depend. Done as $S$ is lin indep spanning set & \therefore a basis.

Step 3: If yes, delete $v$ from $S$ & note by remark that we still have $\text{Span}(S) = V$. Go back to step 1.

Note, process stops \therefore $S$ has only finitely many vectors and you can’t keep deleting forever.

**Dimension**

**Lemma** Let $v_1, \ldots, v_m \in F^n$ be lin indep. Then $m \leq n$.

Proof: Let $A = (v_1 \ldots v_m) \in M_{nm}(F)$.

$v_1, \ldots, v_m$ lin indep

$x_1 v_1 + \ldots + x_m v_m = 0$

has unique soln $0 = x_1 = \ldots = x_m$.

i.e. $A \mathbf{x} = \mathbf{0}$ has unique soln $\mathbf{x} = \mathbf{0}$.

\therefore no. rows of $A \geq$ no. columns of $A$.

(Otherwise row echelon form for $A$ has non-leading
column so gen soln for $x$ has a parameter in it, a contradiction.

$\therefore n \geq m$.

**Thm 3** Let $V = \text{vector space}/ \text{field } \mathbb{F}$

Let $S = \{w_1, \ldots, w_n\}$ span $V$.

Let $I = \{v_1, \ldots, v_m\} \subseteq V$ be lin indep.

Then $m \leq n$.

Proof: By thm 2, we can shrink $S$ to basis $B$ with $d \leq n$ vectors.

Scholium in lecture 10 $\implies$

$[v_1]_B, \ldots, [v_m]_B \in \mathbb{F}^d$ are lin indep.

Lemma $\implies m \leq d$ so $m \leq n$ too.

**Prop - Defn** Let $V = \text{vector space}/ \text{field } \mathbb{F}$

Suppose $B_1 = \{v_1, \ldots, v_m\}$ &

$B_2 = \{w_1, \ldots, w_n\}$ are bases for $V$. Then $m = n$

In this case, say $V$ is finite dimensional
& has dimension $\dim_{\mathbb{F}} V = \dim V := n$.

Proof: By thm 3

$B_1$ spans $V$ & $B_2$ lin indep $\implies$  
$B_2$ spans $V$ & $B_1$ lin indep $\implies$

so $m = n$.

**e.g. 2** For $v \in \mathbb{R}^3 - \mathbf{0}$, the line 

$V = \text{Span}(v)$ has basis $B = \{v\}$.

$\therefore \dim_{\mathbb{R}} V = \text{no. elts of } B = 1$.

So a line is 1-dimensional as one would want.

**e.g. 3** We have standard basis 

$\{e_1, \ldots, e_n\}$ so $\dim \mathbb{F}^n = n$.

**e.g. 4** $\mathbb{C}$ is a vect space / $\mathbb{R}$ with 

vector addn = addn of complex numbers 

scalar multn = product of complex number by real. 

$\mathbb{C}$ has basis (over $\mathbb{R}$)

since every complex number can be written uniquely
as $a + bi$ for some (unique choice of) $a, b \in \mathbb{R}$.

Hence $\text{dim}_{\mathbb{R}} \mathbb{C} =$

Note $\text{dim}_{\mathbb{C}} \mathbb{C} = \text{dim}_{\mathbb{C}} \mathbb{C}^1 = 1$.

**E.g. 5** $\dim M_{mn}(\mathbb{F}) = mn$ since

$\{E_{11}, \ldots, E_{1m}, E_{21}, \ldots, E_{m1}, \ldots, E_{mn}\}$ is a basis.

$\dim \mathbb{P}_n$

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**Theoretical implications of dimension**

**Cor 1** For $V = \text{vector space/ field } \mathbb{F}$ of dim $d$,

a) Any lin indep set $I$ has $\leq d$ elts.

b) Any spanning set $S$ has $\geq d$ elts.

Proof: a) $V$ has a basis $S$ with $d$ elts. $S$ is also a spanning set so

$\text{thm 3 } \implies d \geq \text{no. elts of } I$. 


b) Sim.

**e.g. 5** \( \dim \mathbb{R}^3 = 3 \) so can’t span \( \mathbb{R}^3 \) with \( < 3 \) vectors and any 4 vectors in \( \mathbb{R}^3 \) are lin dependent.

**Cor 2** Let \( V = \) vector space/ field \( \mathbb{F} \). Let \( B = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \subset V \) (\( \mathbf{v}_i \)’s distinct). The following conditions on \( B \) are equivalent:

a) \( B \) is a basis for \( V \).

b) \( B \) is lin indep & \( n = \dim V \).

c) \( B \) spans \( V \) & \( n = \dim V \).

Proof: Omitted, see notes §7.6, thm 3.

**e.g. 6** Let \( B = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \) be an o/n set of vectors in \( \mathbb{R}^n \).

We saw lect 10 that \( B \) is lin indep so cor 2 \( \Rightarrow \) \( B \) is in fact a basis.

**Existence of basis**

**Thm 4** Let \( V = \) vector space/ field \( \mathbb{F} \)
Let $S \subset V$ be a spanning set of $n$ elts. Any subspace $W$ of $V$ has a basis with $\leq n$ vectors.

If $\dim W = \dim V$ then $W = V$.

Proof: Omitted. It’s dual to proof thm 2.

e.g. 7 One can check closure axioms to see $W := \{x \in \mathbb{R}^5 | x_1 + 2x_2 - x_5 = 0\}$ is a subspace of $\mathbb{R}^5$.

Thm 4 $\Rightarrow$ it has a basis with $\leq \dim \mathbb{R}^5 = 5$ vectors. In fact, since $W \neq \mathbb{R}^5$, any basis has $< 5$ vectors.