Lecture 10: Bases & Coordinates

**Aim Lecture** Show how to set up coord systems using bases.

**Basis**

**Defn 1** Let $V = \text{vect space} / \text{field } \mathbb{F}$. $B = \{v_1, \ldots, v_n\} \subset V$ is a basis for $V$ if

1) $\text{Span}(B) = V$

& 2) $B$ is lin indep.

This is equiv to

$(\ast)$ any $v \in V$ can be written uniquely as a lin combn of $B$.

Why? Thm lect 9 says, if $B$ is a basis, then any $v \in \text{Span}(B) = V$ can be written uniquely as a lin combn of $B$.

Conversely, $(\ast) \implies \text{Span}(B) = V$
& the unique way of writing $0$ as a lin combn of $B$ is

$$0 = 0v_1 + \ldots + 0v_n$$

i.e. $B$ is also lin indep. $\therefore B$ is a basis.

e.g. 1 $V = 0$ has basis $\emptyset$.

e.g. 2 $\mathbb{P}_n$ has basis $B = \{1, x, x^2, \ldots, x^n\}$

Why? $B$ spans $\mathbb{P}_n$ $\because \lambda_0 + \lambda_1x + \ldots + \lambda_nx^n \in \text{Span}(B)$

$$& 0 = \lambda_01 + \lambda_1x + \ldots + \lambda_nx^n$$

$$\implies \lambda_0 = \lambda_1 = \ldots = \lambda_n = 0$$

so $B$ is also lin indep. Hence $B$ is a basis.

e.g. 3 $V = M_{mn}(\mathbb{F})$.

For $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$

define $E_{ij} =$
i.e. 0 everywhere but in \((i, j)\)-th entry which is 1. 

\[ B = \{ E_{ij} \} \] is a basis. e.g. for \( M_{22} \) we can uniquely express

**Linearity of coordinates**

Let \( B = \{ v_1, \ldots, v_n \} \) be an ordered basis for vector space \( V / \mathbb{F} \).

**Rem** Then any \( v \in V \) has coord 

\((x_1, \ldots, x_n)\)

where, \( v = x_1 v_1 + \ldots + x_n v_n \).

**Thm 1** For \( u, w, \in V, \alpha \in \mathbb{F} \):

a. \( [u + w]_B = [u]_B + [w]_B \).
b. \([\alpha \mathbf{u}]_B = \alpha [\mathbf{u}]_B\).

N.B. 1. This means fn \(V \rightarrow \mathbb{F}^n : \mathbf{u} \mapsto [\mathbf{u}]_B\) is linear in language of ch.8.

2. Coord allow you to “identify” \(V\) with \(\mathbb{F}^n\) the same way it allows you to “identify” 3-dim space with \(\mathbb{R}^3\).

Proof: Suppose \(\mathbf{u} = u_1 \mathbf{v}_1 + \ldots + u_n \mathbf{v}_n\)

\(\mathbf{w} = w_1 \mathbf{v}_1 + \ldots + w_n \mathbf{v}_n\)

a) \([\mathbf{u} + \mathbf{w}]_B = \)

\(= (u_1 + w_1, \ldots, u_n + w_n)\)

b) Sim.

**Scholium** \(S = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\} \in V\) is lin indep iff \([S]_B := \{[\mathbf{w}_1]_B, \ldots, [\mathbf{w}_m]_B\}\) i.e. can check lin indep on coords.
Proof: (\[\implies\] only). Suppose
\[\lambda_1[w_1]_B + \ldots + \lambda_m[w_m]_B = 0\]
so thm 1 \[\implies\] \([\lambda_1 w_1 + \ldots + \lambda_m w_m]_B = 0\].
Only the the zero vector has coords zero so
\[\lambda_1 w_1 + \ldots + \lambda_m w_m = 0.\]
S lin indep \[\implies\] \(0 = \lambda_1 = \ldots = \lambda_m\)
so \([w_1]_B, \ldots, [w_m]_B\} is lin indep too.

Example

e.g.4 Let \(B = \{v_1, v_2, v_3\}\) where
\[v_1 = (1, 0, 1)^T, v_2 = (0, 1, 0)^T, v_3 = (-1, 1, 1)^T.\]
i) Show \(B\) is a basis for \(\mathbb{R}^3\).
ii) Find the coords \([v]_B\) where \(v = (1, 2, 3)^T\).

A i) Requires showing that for any \(b \in \mathbb{R}^3\), there’s
a unique soln to
\[x_1 v_1 + x_2 v_2 + x_3 v_3 = b.\]
ii) Requires solving
\[ x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_3 = (1, 2, 3)^T. \]

We re-write this lin eqn in matrix form

Solving shows \([\mathbf{v}]_B = \mathbf{x} = \)

To answer i), note that the above computation shows that the row-echelon form for the coeff matrix has all 3 columns leading so back substitution always gives a unique soln regardless of what vector \( \mathbf{b} \) we augment the matrix with. Thus \( B \) is a basis.

**e.g.4 cont’d** Find the vector \( \mathbf{w} \) with coords \((0, 1, 2)^T\) wrt \( B \).

\[
\begin{align*}
\mathbf{A} \cdot \mathbf{w} &= 0 \mathbf{v}_1 + 1 \mathbf{v}_2 + 2 \mathbf{v}_3 \\
&= (0, 1, 0)^T + 2(-1, 1, 1)^T = 
\end{align*}
\]
Orthonormal bases

The most useful coord systems have orthogonal axes.

E.g. In $\mathbb{R}^3$

Recall $B \subset \mathbb{R}^m$ is orthonormal (o/n) if the vectors are unit length & pairwise orthogonal.

**Propn 1** An o/n set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is lin indep

Proof: If $\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n = \mathbf{0}$

then for any $i$ we have

$0 = \mathbf{v}_i \cdot \mathbf{0} = \mathbf{v}_i \cdot (\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n)$

$= \lambda_1 \mathbf{v}_i \cdot \mathbf{v}_1 + \ldots + \lambda_i \mathbf{v}_i \cdot \mathbf{v}_i + \ldots + \lambda_n \mathbf{v}_i \cdot \mathbf{v}_n$

$= \lambda_i \mathbf{v}_i \cdot \mathbf{v}_i = \lambda_i.$
Hence all the scalars $\lambda_i = 0$ & the o/n set must be lin indep.

**Propn 2** If $B = \{v_1, \ldots, v_n\}$ is an o/n basis of $\mathbb{R}^m$ (see later $m = n$) then

$$[v]_B = (x_1, \ldots, x_n)^T$$
where $x_i = v \cdot v_i$.

Proof: None. This is ex 66 of §7.7 of the notes.

Geometric picture is more enlightening.

Suppose $B = \{v_1, v_2\} \subset \mathbb{R}^2$ is o/n.

Alternate characterisation of linear depend

e.g. 5 Suppose non-trivial lin reln holds

$$2v_1 + 2v_3 - 4v_4 = 0.$$
Then we can write each of $v_1, v_2, v_4$ as a lin combn of the others

e.g. $v_1 = 

**Prop 3** i) If $S = \{v_1, \ldots, v_n\}$ is lin depend then there’s some $i$ s.t. $v_i$ is a lin combn of the others.

Conversely, if $v_i$ is a lin combn of the others i.e. $v_i \in \text{Span}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ then $S$ is lin indep.

Proof: just as in above e.g.

( $\implies$ ) Suppose $S$ lin depend so have non-trivial lin reln

$$\lambda_1 v_1 + \ldots + \lambda_n v_n = 0.$$ 

Pick $i$ with $\lambda_i \neq 0$. Such an $i$ exists since reln is non-trivial. Now re-write $v_i$ in terms of others

$$v_i = -\frac{1}{\lambda_i}(\lambda_1 v_1 + \ldots \lambda_{i-1} v_{i-1} + \lambda_{i+1} v_{i+1} \ldots + \lambda_n v_n).$$
Conversely if
\[ \mathbf{v}_i = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_{i-1} \mathbf{v}_{i-1} + \lambda_{i+1} \mathbf{v}_{i+1} + \ldots + \lambda_n \mathbf{v}_n \]
then we can re-arrange to get non-trivial reln
\[ \lambda_1 \mathbf{v}_1 + \ldots + \lambda_{i-1} \mathbf{v}_{i-1} - \mathbf{v}_i + \lambda_{i+1} \mathbf{v}_{i+1} + \ldots + \lambda_n \mathbf{v}_n = 0 \]
so \( S \) is lin depend.