Lecture 1: Complex Number Basics

**Aim Lecture** Extend the real number system to complex number system which includes a square root of -1 denoted $i$.

**Complex numbers**

(Crash course only, see notes for details). We won’t define complex numbers. For us, a complex number $z$ will be a number of the form $a + bi$ where $a, b \in \mathbb{R}$ and $i$ is a new number s.t. $i^2 = 1$.

The expression $a + bi$ is called the *Cartesian form* for $z$. The set of complex numbers is denoted $\mathbb{C}$.

We can $+$, $-$, $\times$ complex numbers to get a complex number. Below $a, a', b, b' \in \mathbb{R}$.

**Add** $(1 + 2i) + (3 + 5i) = 4 + 7i$.

In gen, $(a + bi) + (a' + b'i) =$
**Subtrn** \((1 + 2i) - (3 + 5i) = -2 - 3i.\)

In gen, \((a + bi) - (a' + b'i) = \)

**Multn** \((1 + 2i)(3 + 4i) = 1(3 + 4i) + 2i(3 + 4i) = 3 + 4i + 6i + 8i^2 = 3 + 10i - 8 = -5 + 10i.\)

In gen, \((a + bi)(a' + b'i) = \)

**Rem** (for former MATH1081 students) The best defn of a complex number is an equivalence class of real polynomials in \(i\) where \(p(i) \sim q(i) \text{ iff } (i^2 + 1)|p(i) - q(i).\)

**How’s \(\mathbb{C}\) a number system?**

Any \(x, y, z \in \mathbb{C}\) obey following standard laws of arithmetic.

1. **Associative Laws:**
   \[(x + y) + z = x + (y + z), \quad (xy)z = \]

2. **Commutative Laws:**
\( x + y = y + x, \ xy \)

3. Distributive Law:
\( x(y + z) = xy + xz \)

**Rem** 1. \( \implies \) it doesn’t matter how you bracket if you stick to just adding or just multiplying. You need the brackets when you add and multiply.

Real and imaginary parts

**Defn** Let \( z = a + bi \in \mathbb{C}, a, b \in \mathbb{R} \).

Its real part is \( \text{Re } z = a \).

Its imaginary part is \( \text{Im } z = \)

**e.g.** \( \text{Re } 4 - 5i = \)

**Thm** Let \( a, a', b, b' \in \mathbb{R} \). If \( a + bi = a' + b'i \) then \( a = a', b = b' \) i.e. complex numbers are uniquely determined by their real & imaginary parts.

Proof: \( a - a' = b'i - bi = \)
\[(a - a')^2 = \]

\[\therefore a = a', b = b'.\]

**Conjugation & Division**

**Defn** Let \( z = a + bi \in \mathbb{C}, a, b \in \mathbb{R}. \)

Its complex conjugate is \( \bar{z} = a - bi. \)

**e.g.** \( 5 - i = \)

**Formula** \( z\bar{z} = (a + bi)(a - bi) = a^2 - b^2i^2 \)

so \( z\bar{z} = a^2 + b^2. \)

This is real!!

**Division** of complex numbers. Trick is to multiply top & bottom by conjugate of the denominator as follows.

\[
\frac{1+2i}{3+4i} = \frac{1+2i(3-4i)}{3+4i(3-4i)} = \frac{(1+2i)(3-4i)}{(3+4i)(3-4i)}
\]

\[
= \frac{3+6i-4i-8i^2}{3^2+4^2} = \frac{11+2i}{25} =
\]
Properties of conjugation

**Formulae** For \( w, z \in \mathbb{C} \),

1. \( \overline{\bar{z}} = z \)

2. \( \overline{z - w} = \bar{z} - \bar{w}, \quad \overline{z + w} = \)

3. \( \overline{zw} = \bar{z}\bar{w}, \)

4. \( z \cdot w = \bar{z}\bar{w}, \)

5. \( \text{Re } z = \frac{1}{2}(z + \bar{z}), \quad \text{Im } z = \frac{1}{2i}(z - \bar{z}) \)

Proof: easy exercise using Cartesian forms e.g. for 5, if \( z = a + bi, \ a, b \in \mathbb{R} \), then

\[
\frac{1}{2}(z + \bar{z}) =
\]

**Defn** A complex number \( z \) is real if \( \text{Im } z = 0 \) i.e. by 5. above, \( z = \bar{z} \). It is purely imaginary if \( \text{Re } z = 0 \) i.e.

**e.g.** Show that \( u = \bar{z}w + z\bar{w} \) is real.

**A** \( \bar{u} = \)


How’s \( \mathbb{C} \) unlike the real number system?

The set \( P \) of positive real numbers can be used to order \( \mathbb{R} \). \( P \) satisfies the following.

i) Any \( x \in \mathbb{R} \) satisfies exactly one of the following:
   a) \( x = 0 \) OR b) \( x \in P \) OR c) \( -x \in P \).

ii) \( P \) is closed under addition i.e.
    for any \( x, y \in P \) we also have \( x + y \in P \).

iii) \( P \) is closed under multiplication i.e.
    for any \( x, y \in P \) we also have

We cannot order \( \mathbb{C} \) the same way for suppose we can find \( P \subset \mathbb{C} \) s.t. ii),iii) hold and i) holds with \( \mathbb{R} \) replaced with \( \mathbb{C} \). Then either
Argand diagram. Polar form.

Represent \( z = a + bi \in \mathbb{C}, a, b \in \mathbb{R} \) by point in plane with co-ordinates \((a, b)\). Above plane called the complex plane & the axes are the real & imaginary axes.
Polar coords on plane suggests

**Defn** The modulus of \( z = a + bi \), \( a, b \in \mathbb{R} \) is

\[ |z| := \text{distance } r \text{ from } 0 \text{ to } z \]

The argument of \( z(\neq 0) \) is the angle

\[ \text{Arg } z = \theta \in (-\pi, \pi] \text{ in picture so } \tan \theta = \]

Note \( \theta \) is measured anti-clockwise from the positive real axis so is negative if \( z \) lies below the real axis.

**e.g.**

\[ |3 + 4i| = \]
\[ \text{Arg } 3 + 4i = \]
\[ | -3 - 4i| = \]
\[ \text{Arg} -3 - 4i = \]

Answer here NOT \( \tan^{-1}\left(\frac{-4}{-3}\right) \).

e.g. \( |\bar{z}| = \)

\( \text{Arg} \bar{z} = - \text{Arg} z \) unless

**Polar form** Consider Cartesian form \( z = a + bi \).

If \( r = |z|, \theta = \text{Arg} z \) then

\( \cos \theta = \)

\[ \implies z = a + bi = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta) \]

This is called the polar form of \( z \).

e.g. Write \( z = 1 + \sqrt{3} \) in polar form.

\( |z| = \)

\( z \) is in the 1st quadrant, so \( \text{Arg} z = \tan^{-1} \sqrt{3} = \)

The polar form is \( z = \)