Lecture 7: Vector Spaces

**Aim Lecture** Introduce vector spaces which provide natural

**Motivation:** some linear phenomena

**In 3D-space**

Line has param form

**In** \( \mathbb{R}^3 \): 2 inequivalent non-

**Function space** Soln to \( \frac{dy}{dx} = 2x \) is

“line”
Graphically get

Wish to define notion of vector space where

More gen, want standard rules

**Vector space axioms**

Let $F$ be a field like $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$. A vector space over $F$ is

- a. A set $V$ of
- & b. An addition law denoted $+$ which

This new vector is called

- & c. A scalar multn law
such that the following

For any $u, v, w \in$

1. Associative Law of Addition:

$$(u + v) + w =$$

2. Commutative Law of Addition:

$$u + v =$$

3. Existence of Zero: there’s a vector called

4. Existence of Negatives: there’s a vector called

5. Associative Law for Scalar Multn:

$$(\lambda \mu) v =$$

6. $1v =$

7. Scalar Distributive:
\[(\lambda + \mu)v = \]

8. Vector Distributive:
\[\lambda(v + w) = \]

**Examples**

*Example 1* \(\mathbb{R}^n\) is a vector space over

addition rule:

scalar multiplication rule:

*Example 2* \(\mathbb{C}^n\) is a vector space over

define addn law =

scalar multn =

*Example 3* Let \(V\) = set of geom
(i.e. “arrows”) in
$V$ is a vector space
$\text{addn} =$
scalar multn

**e.g. 4** Let $F = \mathbb{R}$ or $\mathbb{C}$ or any other field.

Let $M_{mn}(F)$ be set of $m \times n$-matrices over $F$.

Define

vector addition to be
scalar multiplication by

You can check all axioms. Here we’ll only check axiom 4:
e.g. 5 $X = $ non-empty set.
$\mathcal{R}[X] := $ set of real-valued
$\mathcal{R}[X]$ is a vector space / $\mathbb{R}$ if define
vector addn: for $f, g \in$
$(f + g)$
scalar multn: for $\lambda \in \mathbb{R}, f \in$
$\lambda f$
Then $\mathcal{R}[X]$ is a vector space / $\mathbb{R}$ with zero
the
Sim e.g. 6 $X = $ non-empty set.
$\mathcal{C}[X] := $ set of
$\mathcal{C}[X]$ is a vector space / $\mathbb{C}$ if define
addn & scalar multn point

Properties of vector spaces
Let $V = \text{vector space} / \text{field } \mathbb{F}$.

**Subtraction** Let $v, w \in V$. $-v$ is the only vector s.t.

We can define $w -$

Standard rules of vector

**Example.** For $v, w \in V$ simplify

$2(3v + 4w) +$

**Proposition** Let $V$ be a vector space / $\mathbb{F}$. For $\lambda \in \mathbb{F}, v \in V$

1. $\lambda 0 =$
2. $0v =$
3. $(-1)v =$
4. $\lambda \mathbf{v} = 0 \implies$

Proof: $0 \mathbf{v} + 0 \mathbf{v} =$

Subtract

$0 \mathbf{v} + 0 \mathbf{v} - 0 \mathbf{v} =$

So $0 \mathbf{v} =$

1) is sim.

3) $(-1) \mathbf{v} +$

4) If $\lambda \neq 0$ then

$\mathbf{v} = 1 \mathbf{v} =$

An exotic example

**e.g.** Twisted $\mathbb{C}^n$. $V =$

Addn:
New twisted scalar multn:

$V$ is a vector

Why?