Lecture 7: Vector Spaces

Aim Lecture Introduce vector spaces which provide natural

Motivation: some linear phenomena

In 3D-space

Line has

param form

In $\mathbb{R}^3$**:** 2 inequivalent non-

Function space Soln to $\frac{dy}{dx} = 2x$

is

"line"

Graphically get

Wish to define notion of vector space where

More gen, want standard rules

Vector space axioms

Let $\mathbb{F}$ be a field like $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. A vector space over $\mathbb{F}$ is

a. A set V of

& b. An addition law denoted $+$ which

This new vector is called

& c. A scalar multn law

such that the following

For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in$

1. Associative Law of Addition:

$(\mathbf{u} + \mathbf{v}) + \mathbf{w} =$

2. Commutative Law of Addition:

$\mathbf{u} + \mathbf{v} =$

3. Existence of Zero: there's a vector called

4. Existence of Negatives: there's a vector called

5. Associative Law for Scalar Multn:

$(\lambda\mu)\, \mathbf{v} =$

6. $1\mathbf{v} =$

7. Scalar Distributive:

$(\lambda + \mu)\,\mathbf{v} =$

8. Vector Distributive:

$\lambda(\mathbf{v} + \mathbf{w}) =$

Examples

e.g. 1 $\mathbb{R}^n$ is a vector space over

addition rule:

scalar multiplication rule:

e.g. 2 $\mathbb{C}^n$ is a vector space over

define addn law $=$

scalar multn $=$

e.g. 3 Let $V =$ set of geom

(i.e. “arrows”) in

V is a vector space

addn =

scalar multn

e.g. 4 Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ or any other field. Let $M_{mn}(\mathbb{F})$ be set of $m \times n$-matrices over $\mathbb{F}$.

Define

vector addition to be

scalar multiplication by

You can check all axioms. Here we'll only check axiom 4:

e.g. 5 X = non-empty set.

$\mathcal{R}[X] :=$ set of real-valued

$\mathcal{R}[X]$ is a vector space / $\mathbb{R}$ if define

vector addn: for $f, g \in$

$(f + g)$

scalar multn: for $\lambda \in \mathbb{R}, f \in$

λf

Then $\mathcal{R}[X]$ is a vector space / $\mathbb{R}$ with zero

the

Sim **e.g. 6** X = non-empty set.

$\mathcal{C}[X] :=$ set of

$\mathcal{C}[X]$ is a vector space / $\mathbb{C}$ if define

addn & scalar multn point

Properties of vector spaces

Let V = vector space / field $\mathbb{F}$.

Subtraction Let $\mathbf{v}, \mathbf{w} \in V$. $-\mathbf{v}$ is the only vector s.t.

We can define $\mathbf{w} -$

Standard rules of vector

e.g. 8 For $\mathbf{v}, \mathbf{w} \in V$ simplify

$2(3\,\mathbf{v} + 4\,\mathbf{w}) +$

Propn Let V be a vector space / $\mathbb{F}$. For $\lambda \in \mathbb{F}, \mathbf{v} \in V$

1. $\lambda\mathbf{0} =$ 2. $0\,\mathbf{v} =$ 3. $(-1)\,\mathbf{v} =$

4. $\lambda\,\mathbf{v} = \mathbf{0} \Longrightarrow$

Proof: $0\,\mathbf{v} + 0\,\mathbf{v} =$

Subtract

$0\,\mathbf{v} + 0\,\mathbf{v} - 0\,\mathbf{v} =$

So $0\,\mathbf{v} =$

1) is sim.

3) $(-1)\,\mathbf{v} +$

4) If $\lambda \neq 0$ then

$\mathbf{v} = 1\,\mathbf{v} =$

An exotic example

e.g. Twisted $\mathbb{C}^n$. $V =$

Addn:

New twisted scalar multn:

V is a vector

Why?