Lecture 24: Conics. Symmetric matrices.

**Aim lecture** See how diagonalising

## Quadratic forms

**Defn 1** A (real) quadratic form in $n$ variables is a fn of form

$Q(x_1, \ldots, x_n) =$

for some $a_{ij} \in$

The set of solns to $Q(x_1, x_2) = 1$ where $Q$

**e.g. 1** $\frac{1}{a^2}x_1^2+$

**e.g. 2** $\frac{1}{a^2}x_1^2-$

**Q** What do solns of $7x_1^2 + 8x_1x_2 + 13$

Ans: at end

## Quadratic forms via symmetric matrices

Put $x_1, \ldots, x_n$ into vector

For $A \in M_{nn}(\mathbb{R})$, the fn $Q_A : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$Q_A(\mathbf{x}) =$

**e.g. 3**

$$(x_1 \quad x_2) \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Conversely,

**Prop 1** Any quadratic form $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ has form

$Q(\mathbf{x}) =$

Proof: Hopefully clear from any e.g.

$Q(x_1, x_2) = 7x_1^2 + 8x_1x_2 +$

Try $A =$

Then $\mathbf{x}^T A \mathbf{x} =$

## Orthogonal matrices

**Defn 2** A square matrix $A \in M_{nn}(\mathbb{R})$ is orthogonal

**N.B.** Orthog matrices are invertible.

**Prop 2** A square matrix $A = (\mathbf{a}_1 \ldots \mathbf{a}_n)$ is orthog iff

Proof: Note that

$$A^T A = \begin{pmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_n^T \end{pmatrix}$$

$$=$$

This is $I_n$ iff

**e.g. 4** The rotation matrix

$R_\theta =$

Check: $R_\theta^T$

Also the columns $\mathbf{a}_1, \mathbf{a}_2$ of $R_\theta$ are

## Diagonalising symmetric matrices

**Thm** Let $A \in M_{nn}(\mathbb{R})$ be

i.e. $A^T = A$. Then

1) the e-values of $A$

2) $A$ is diag & we can pick an e-basis $B = \{\mathbf{f}_1, \dots$

Proof: In case $n = 2$ only.

1) Suppose $A =$

Its char

The discriminant of this is

so the roots

This proves 1)

2) Let $\mathbf{f}_1 \in \mathbb{R}^2$ be an e-vector with

Scale so

Let $\mathbf{f}_2$ be a vector

& unit

It suffices to show $\mathbf{f}_2$ is

$(A\,\mathbf{f}_2)$

so $A\,\mathbf{f}_2$ is also

$\therefore A\,\mathbf{f}_2$ is a scalar

i.e. $\mathbf{f}_2$ is also an

**e.g. 5** Diagonalise

$$A = \begin{pmatrix} 7 & 4 \\ 4 & 13 \end{pmatrix}$$

**A** The char poly is

Note e-values are

$\lambda = 5$ e-space:

$\lambda =$

So we get an o/n

## Application to conics

**Cor** Let $A$ be

& consider quadratic form $Q(\mathbf{x}) =$

Let $M = (\mathbf{f}_1$

$B = \{\mathbf{f}_1, \ldots$

& $\lambda_1, \ldots, \lambda_n$ are the

If we change variables to $\mathbf{y} =$

$Q$ becomes

$Q(\mathbf{x}) = \lambda_1 y_1^2 +$

Proof: Let $D$ be the diagonal matrix with diag entries so that $A = M$

Then $Q(\mathbf{x}) = \mathbf{x}^T A\,\mathbf{x} = \mathbf{x}^T M$

$= \mathbf{x}^T (M^{-1})^T$

$= \mathbf{y}^T$

$= (y_1 \ \ldots \ y_n)$

$= (y_1 \ \ldots \ y_n)$

$= \lambda_1 y_1^2 +$

**e.g.5 again** Describe geometrically

$7x_1^2 + 8x_1x_2 + 13x_2^2 = 1$

**A** We saw this is equivalent to

$(*) \qquad Q(\mathbf{x}) = \mathbf{x}^T A\,\mathbf{x} =$

where $A =$

From e.g. 5 we saw the e-values & e-vectors were

So if we change coords to $y_1, y_2$ wrt

(*) becomes