Lecture 22: Powers of Matices

Aim lecture See applications of diagonalisation to

Geometric interpretation of powers

E.g. 1 Consider $T = T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with e-basis $B =$

$T^2\,\mathbf{v}_1 =$

$T^3\,\mathbf{v}_1 =$

$T^k \mathbf{v}_1 =$

It's thus easy to compute powers of T wrt

If D repr

then T^k (wrt B) is repr by

Change of basis result of thm 1 lect 19 $\Longrightarrow$

$A^k =$

Let's repeat this problem algebraically as opposed to geom.

Powers of matrices

Lemma The product of diagonal matrices is

Proof: Easy computation.

Prop 1) Consider the diagonal matrix

$D =$

Then $D^k =$

2) If $A = MDM^{-1}$ then

$A^k =$

Proof: 1) by

2) $A^k =$

$=$

e.g. 2 In e.g. 1 of lecture 20 saw

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}$$

Find A^5.

Example of decoupled dts

e.g. 3 Let

$x_1(k) =$ popn of elves

$x_2(k) =$

Suppose popn dynamics governed by recursion reln

$x_1(k+1) = 3x_1(k)$

$x_2(k+1) = 2$

Soln is

Can rewrite 2 eqns as

$\mathbf{x}(k+$

where $\mathbf{x}$

Q What if A is not diagonal?

Diagonalisation & decoupling dts

Let $\mathbf{x}(k) =$

Consider the following recurrence reln

(∗) $\mathbf{x}(k+1) =$

where $A \in$

This is another example of a

Note $\mathbf{x}(1) =$

$\mathbf{x}(2) =$

$\mathbf{x}(k) =$

Conclusion Can solve (*) by computing powers of matrices as in e.g. 2.

Alternatively, repeat analysis for cts to get

Thm Suppose $A = MD$

where $M = (\mathbf{f}_1$

The soln to $\mathbf{x}(k+$

$(\dagger)\ \mathbf{x}(k) = \alpha_1 \lambda$

Proof: As in lect 21 or directly by induction on

$k = 0$: Pick $\alpha_1, \ldots$

$\mathbf{x}($

which is possible since $\{\mathbf{f}_1$

$k > 0$: Suppose $(\dagger)$

$\mathbf{x}(k+1) = A(\alpha_1$

$= \alpha_1 \lambda_1^k A$

$=$

Example of dts

e.g. 4 Solve $\mathbf{x}(k$

$$A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$$

A Same matrix as in e.g. 1 lect 21 so $A = MDM^{-1}$

$D =$

$\therefore \mathbf{x}(k) =$

Second order dts

As for cts, can convert 2nd order difference eqn into 2 var recurrence reln.

e.g. 4 Solve $x(k+2) - 3x(k+1) + 2$

Ans: Let $x_1(k) =$

$x_2(k) =$

$\therefore x_1(k+$

$x_2(k+$

$\mathbf{x}(k+$

same matrix as in e.g. 4 lect 22 so $A = M$

$\therefore \mathbf{x}(k) =$

and $x(k) =$

N.B. Characteristic poly of A corresponds to