Lecture 22: Powers of Matrices

**Aim lecture** See applications of diagonalisation to

**Geometric interpretation of powers**

**E.g. 1** Consider $T = T_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with e-basis $B = \ldots$

\[ T^2 v_1 = \ldots \]
\[ T^3 v_1 = \ldots \]
$T^k \mathbf{v}_1 =$

It’s thus easy to compute powers of $T$ wrt $D$ repr

then $T^k$ (wrt $B$) is repr by

Change of basis result of thm 1 lect 19 $\implies$

$A^k =$

Let’s repeat this problem algebraically as opposed to geom.

Powers of matrices

**Lemma** The product of diagonal matrices is
Proof: Easy computation.

Prop 1) Consider the diagonal matrix

\[ D = \]

Then \[ D^k = \]

2) If \( A = MDM^{-1} \) then

\[ A^k = \]

Proof: 1) by

2) \( A^k = \)

\[ = \]

e.g. 2 In e.g. 1 of lecture 20 saw

\[ A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \]
Find $A^5$.

Example of decoupled dts

e.g. 3 Let

$x_1(k) = \text{popn of elves}$
\[ x_2(k) = \]
Suppose popn dynamics governed by recursion reln
\[ x_1(k + 1) = 3x_1(k) \]
\[ x_2(k + 1) = 2 \]
Soln is

Can rewrite 2 eqns as
\[ \mathbf{x}(k+) \]
where \( \mathbf{x} \)

Q What if \( A \) is not diagonal?

Diagonalisation & decoupling dts

Let \( \mathbf{x}(k) = \)
Consider the following recurrence reln

\[ (*) \]

\[ \mathbf{x}(k + 1) = \]

where \( A \in \)

This is another example of a

Note \( \mathbf{x}(1) = \)

\( \mathbf{x}(2) = \)

\( \mathbf{x}(k) = \)

**Conclusion** Can solve \((*)\) by computing powers of matrices as in e.g. 2.

Alternatively, repeat analysis for cts to get

**Thm** Suppose \( A = MD \)

where \( M = (f_1 \)


The soln to $x(k+1)$

(†) $x(k) = \alpha_1 \lambda^k$

Proof: As in lect 21 or directly by induction on

$k = 0$: Pick $\alpha_1, \ldots$

$x(0)$

which is possible since $\{f_1, k > 0\}$

$k > 0$: Suppose (†)

$x(k + 1) = A(\alpha_1$

$= \alpha_1 \lambda^k A$

Example of dts

7
e.g. 4 Solve $x(k)$

$$A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$$

The same matrix as in e.g. 1 lect 21 so $A = MDM^{-1}$

$D =$

$\therefore x(k) =$

Second order dts

As for cts, can convert 2nd order difference eqn into 2 var recurrence reln.

e.g. 4 Solve $x(k + 2) - 3x(k + 1) + 2$

Ans: Let $x_1(k) =$
\[ x_2(k) = \]
\[ \therefore x_1(k+ \]
\[ x_2(k+ \]
\[ \mathbf{x}(k+ \]

same matrix as in e.g. 4 lect 22 so \( A = M \)

\[ \therefore \mathbf{x}(k) = \]

and \( x(k) = \)

N.B. Characteristic poly of \( A \) corresponds to