Lecture 21: DEs and diagonalisation

Aim lecture See how e-vectors

Motivation

e.g. 1 $y_1(t)$ = popn of hobbits in

$y_2(t)$ = popn of orcs

If 2 popn kept separate as here then popn growth governed by a pair of DEs which typically looks something like:

$$\begin{aligned} y_1'(t) &= 3y_1(t) \\ y_2'(t) &= 2y_2(t) \end{aligned} \qquad (*)$$

Soln: Easy, solve 2 eqns <u>separately</u>

Suppose now we put the two popns in

Typical DEs describing popn growth is

$$\begin{aligned} y_1'(t) &= 3y_1(t) - 2y_2(t) \\ y_2'(t) &= -y_1(t) + 2y_2(t) \end{aligned} \qquad (\dagger)$$

These are "coupled" DEs i.e. y_1', y_2' each depend on both $y_1 \& y_2$. We'll use diag to

Notn $\mathbf{y}(t) =$

$\mathbf{y}'(t) = \frac{d\,\mathbf{y}}{dt} :=$

In e.g. 1, $\mathbf{y}'(t) = A\,\mathbf{y}(t)$ where
$A =$

Note In decoupled case (*) above, still have

$\mathbf{y}'(t) = A\,\mathbf{y}(t)$ but now

Diagonalisation & decoupling DEs

Consider more generally

$\mathbf{y}(t) =$

& system of n linear DEs

$\mathbf{y}'(t) = A\,\mathbf{y}(t)$

where $A \in M_{n,n}(\mathbb{R})$.

Lemma For $C \in M_{n,n}(\mathbb{R})$

$\frac{d}{dt}$

Proof: Clear from case $n = 2$. Say

$C =$

$\frac{d}{dt}(C\,\mathbf{y}) =$

To solve $\mathbf{y}'(t) = A\,\mathbf{y}(t)$, suppose we can diag $A = MDM^{-1}$ with $M = (\mathbf{f}_1$

& $D =$

Thm 1) Wrt e-basis $B = \{$

if $\mathbf{x}(t) = [\mathbf{y}(t)]_B = M^{-1}$

then get decoupled eqn

(*) $\qquad \frac{d\,\mathbf{x}}{dt}$

2) Soln to (*) is

$x_i(t) = \alpha_i$

3) Soln to original DE $\mathbf{y}'(t) = A\,\mathbf{y}(t)$ is

$\mathbf{y}(t) = M\,\mathbf{x}(t) =$

Proof: 1) $\mathbf{y}'(t) = MD$

Thus $\frac{d}{dt}(M\,\mathbf{x}(t)) =$

so $D\,\mathbf{x}(t) = M^{-1}$

by lemma $=$

2) (*) corresponds to system of linear DEs

3) Just multiply matrices.

Example

e.g. 1 completed

We diag A

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ -1 & 2-\lambda \end{vmatrix}$$

The e-values are

E-vectors?

$\lambda = 4 : \ker(A - \lambda I) =$

An e-vector is

$\lambda = 1 : \ker(A - \lambda I) =$

An e-vector is

Thm 3) $\implies$

$\mathbf{y}(t) =$

i.e. $y_1(t) =$

y_2

e.g. 2 Suppose in e.g. 1 that initial popn is $\mathbf{y}(0)^T = (4000, 1000)$. Solve the IVP.

Ans: We need only solve for α_1, α_2.

From Gaussian elim or guessing see

$\alpha_1 =$

The soln is thus

$\mathbf{y}(t) =$

e.g. 3 What happens in e.g. 2 as $t \longrightarrow$

Nasty hobbits

N.B. Key to limiting behaviour is e-

Second order DEs

We can convert any 2nd order ODE into a pair of linear ODEs in 2 var as in following

E.g. 3 Solve IVP

$$y'' - 3y' + 2y = 0 \quad , \quad y(0) = 2, y'(0) = 3$$

Ans: Let $y_1 = y, y_2 =$

$y_1' =$

i.e. $\mathbf{y}' =$

Diag A: $\det(A - \lambda I) =$

Hence e-values are

E-vectors:

$\lambda = 2 : \ker(A - \lambda I) =$

An e-vector is

$\lambda = 1 : \ker(A - \lambda I) =$

An e-vector is

Hence, (from thm 3)) general soln is

$\mathbf{y}(t) =$

Need now find integration constants.