Lecture 21: DEs and diagonalisation

**Aim lecture** See how e-vectors

**Motivation**

e.g. 1 \( y_1(t) = \) popn of hobbits in \\
y_2(t) = \) popn of orcs

If 2 popn kept separate as here then popn growth governed by a pair of DEs which typically looks something like:

\[
\begin{align*}
y'_1(t) &= 3y_1(t) \\
y'_2(t) &= 2y_2(t)
\end{align*}
\]

(*)

Soln: Easy, solve 2 eqns separately

Suppose now we put the two popns in
Typical DEs describing popn growth is

\[ y_1'(t) = 3y_1(t) - 2y_2(t) \]
\[ y_2'(t) = -y_1(t) + 2y_2(t) \] \hspace{1cm} (†)

These are “coupled” DEs i.e. \( y_1', y_2' \) each depend on both \( y_1 & y_2 \). We’ll use diag to

**Notn** \( y(t) = \)

\[ y'(t) = \frac{dy}{dt} := \]

In e.g. 1, \( y'(t) = A \cdot y(t) \) where

\( A = \)

**Note** In decoupled case (*) above, still have
\( y'(t) = A y(t) \) but now

**Diagonalisation & decoupling DEs**

Consider more generally

\[ y(t) = \]

& system of \( n \) linear DEs

\[ y'(t) = A y(t) \]

where \( A \in M_{n,n}(\mathbb{R}) \).

**Lemma** For \( C \in M_{n,n}(\mathbb{R}) \)

\[ \frac{d}{dt} \]

Proof: Clear from case \( n = 2 \). Say

\( C = \)
\[
\frac{d}{dt}(C \mathbf{y}) =
\]

To solve \( \mathbf{y}'(t) = A \mathbf{y}(t) \), suppose we can \( \text{diag } A = MDM^{-1} \) with \( M = (f_1 \& D) \)

\( B \)

Then get decoupled eqn

**Thm 1)** Wrt e-basis \( B = \{ \)

if \( \mathbf{x}(t) = [\mathbf{y}(t)]_B = M^{-1} \)

then get decoupled eqn
\[(\ast) \quad \frac{d\mathbf{x}}{dt}\]

2) Soln to \((\ast)\) is

\[x_i(t) = \alpha_i\]

3) Soln to original DE \(y'(t) = Ay(t)\) is

\[y(t) = M \mathbf{x}(t) = \]

Proof: 1) \(y'(t) = MD\)

Thus \(\frac{d}{dt}(M \mathbf{x}(t)) = \)

so \(D \mathbf{x}(t) = M^{-1}\)

by lemma =

2) \((\ast)\) corresponds to system of linear DEs

3) Just multiply matrices.

Example
e.g. 1 completed

We diag $A$

$$
\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -2 \\ -1 & 2 - \lambda \end{vmatrix}
$$

The e-values are

E-vectors?

$\lambda = 4 : \ker(A - \lambda I) =$

An e-vector is

$\lambda = 1 : \ker(A - \lambda I) =$

An e-vector is

Thm 3) $\implies$
\[ y(t) = \]

i.e. \( y_1(t) = \)

\( y_2 \)

**e.g. 2** Suppose in e.g. 1 that initial popn is \( y(0)^T = (4000, 1000) \). Solve the IVP.

Ans: We need only solve for \( \alpha_1, \alpha_2 \).

From Gaussian elim or guessing see

\[ \alpha_1 = \]

The soln is thus

\[ y(t) = \]
e.g. 3 What happens in e.g. 2 as $t \rightarrow$

Nasty hobbits

N.B. Key to limiting behaviour is e-

Second order DEs

We can convert any 2nd order ODE into a pair of linear ODEs in 2 var as in following

E.g. 3 Solve IVP

$$y'' - 3y' + 2y = 0 \quad , \quad y(0) = 2, \ y'(0) = 3$$

Ans: Let $y_1 = y, y_2 =$

$$y'_1 =$$

i.e. $y' =$
Diag $A$: $\det(A - \lambda I) =$

Hence e-values are

E-vectors:

$\lambda = 2 : \ker(A - \lambda I) =$

An e-vector is

$\lambda = 1 : \ker(A - \lambda I) =$

An e-vector is

Hence, (from thm 3)) general soln is

$y(t) =$
Need now find integration constants.