Lecture 20: Computing e-bases. Diagonalisation

For linear $T : V \longrightarrow V$

Q 1. How do you find the preferred basis? i.e.

2. How do these bases improve our understanding of T & simplify

Aim (this lecture): Answer Q1.

Aim (next few lectures): Answer

Finding e-values. Characteristic polynomials

Lemma Let $T : V \longrightarrow V$ be a

$\mathbf{v} \in V - \mathbf{0}$ is an e-vector with e-value λ iff $\mathbf{v} \in$

Proof: $\mathbf{v} \neq \mathbf{0}$ is an e-vector of T with

$\iff T\,\mathbf{v} =$

$\iff T\,\mathbf{v} -$

$\iff (T - \lambda$

$\iff \mathbf{v} \in$

Consequence If you know the e-values of T then the e-vectors can be found by

Find e-values using

Prop-Defn Let $A \in M_{n,n}(\mathbb{F})$. The polynomial (in λ)

$p(\lambda) :=$

is called the characteristic

1) $p(\lambda)$ is a poly of degree

2) Its roots are

Proof: 1) Will be clear from

2) Lemma $\implies$ λ is an e-value of A

iff $\ker(A - \lambda I)$ has a

iff ker

iff $A-$

iff det

N.B. To ensure $p(\lambda)$ has roots, often advantageous to work with $\mathbb{F} = \mathbb{C}$.

Example e-basis computation

Defn 1 The λ-eigenspace of a linear map $T : V \longrightarrow V$ is

E.g. 1 Find e-values & e-vectors of

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

Ans:

Step 1: Find roots of $p(\lambda) =$

Step 2: Find corresponding e-vectors

For $\lambda = 2$ e-space:

$\ker(A - \lambda I) = \ker$

$\therefore \alpha\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$

For $\lambda = -1$ e-space:

$\ker(A - \lambda I) =$

$\therefore \alpha\binom{2}{1}$

Diagonalising matrices

Defn $A \in M_{n,n}(\mathbb{F})$ is diagonalisable (over $\mathbb{F}$) if there's a diag matrix

$D =$

& an invertible $n \times n$-matrix M s.t.

Finding such an expression is called diagonalising A.

Thm Let $A \in M_{n,n}(F)$. A can be diagonalised with $A = MDM^{-1}$ iff

i)the columns $B = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ of M form an

& ii) D has diag entries the

Proof: ($\Longleftarrow$) only.

Note M is invertible iff B

The matrix representing $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ wrt B is by thm 2 lect 19 $D =$

& by thm 1 lect 19 M^{-1}

$\therefore A =$

e.g. 2 Diagonalise

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{pmatrix}$$

Note, if A as in e.g. 1, then $B =$

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -4-\lambda & 6 \\ 0 & -3 & 5-\lambda \end{vmatrix} =$$

So e-values

E-vectors:

$\lambda = 1$ e-space:

$$\ker(B - \lambda I) = \ker \begin{pmatrix} & 1 & 0 \\ 0 & & 6 \\ 0 & -3 & \end{pmatrix}$$

An e-vector with e-value 1 is

$\lambda = 2$ e-space:

$$\ker(B - \lambda I) = \ker \begin{pmatrix} & 1 & 0 \\ 0 & & 6 \\ 0 & -3 & \end{pmatrix}$$

An e-vector with e-value 2 is

$\lambda = -1$ e-space:

$$\ker(B - \lambda I) = \ker \begin{pmatrix} & 1 & 0 \\ 0 & & 6 \\ 0 & -3 & \end{pmatrix}$$

An e-vector with e-value -1 is

If $M =$

then $B =$

Existence of e-bases

The question of whether $A \in M_{nn}(\mathbb{F})$ has an e-basis so is

One of the few results we prove in this course is

Prop Let $T : V \longrightarrow V$ be

1) If $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ are e-vectors with distinct

2) If $A \in M_{n,n}(\mathbb{F})$ has n distinct e-values then

Proof: 1) $\implies$ 2) so we only prove 1).

By way of contradiction, suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is

$\alpha_1 \mathbf{v}_1 +$

some α_i

Rearranging indices, we can assume

$\alpha_1 \mathbf{v}_1 +$

with $\alpha_1 \neq 0, \ldots, \alpha_i \neq 0$ & i is

$$\mathbf{0} = (T - \lambda_i I)(\alpha_1 \mathbf{v}_1 +$$

$$= \alpha_1 (T - \lambda_i I) \mathbf{v}_1 +$$