Lecture 19: Change of Basis. Eigenbases

**Inevitable Woffle**

**Q** Why did we introduce abstract notion of vector spaces?

**A** 1. To handle infinite dim
2. Defn is

**Aim Lecture** Show how to simplify some calculations by choosing a sensible

3-dim rotations

e.g. 1 in sensible

Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be rotation
Note: Can show $T$ is
As in lect 16 we see
$T \mathbf{e}_1 =$
$T \mathbf{e}_2 =$
Also $T \mathbf{e}_3 =$
Matrix reprn thm $\implies T$ is multn by $R_{\theta} = (T$

Note: What makes this coord system easy to work with, is that you can deal with the $x_3$ coord
Q What’s the matrix representing rotn about arbitrary

Idea of method Pick sensible o/n basis

\[ B = \{ f_1, f_2, f_3 \} = \]

switch to coords

wrt

rotate using
$R$

change back to

Change of basis

Let $B = \{f_1, \ldots, f_n\}$

Need to know matrix representing $S_B$
Lemma Let $M = (f_1$

Then $S_B(v) := [v]_B =$

Proof: Formula 3 lect 16 $\implies$ we need only show $S_B^{-1}$

$S_B^{-1} e_i =$ vector with

$= \quad \therefore S_B^{-1}$ is matrix muln by

$(S_B^{-1} e_1 \ldots$

e.g. 2 Find matrix $A$ representing rotation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ about axis through $\frac{1}{3}(1, 2, 2)^T$

A Let $B = \{ f_1 = \frac{1}{3}(2, -2, 1), f_2 = \frac{1}{3}(2, 1, \quad ), f_3 = \frac{1}{3}(1, 2, \quad ) \}$.

ex. Check $B$ is an o/n basis which is “right-
Let $M = (f_1$

Then $A =$

More generally,

**Thm 1** Let $B = \{f_1, \ldots f_n\} \subset \mathbb{F}$

& $M = (f_1$

For $A \in M_{nn}(\mathbb{F})$, the matrix $C$ representing

the lin map

Proof: For $C$ to represent $T_A$ wrt $B$ means

$[A \mathbf{v}]_B =$
i.e. $S_B^1$

Lemma $\implies$ this means

$M^{-1}A \mathbf{v} =$

$\therefore M^{-1}$

$\therefore C =$

**e.g. 2 redone** $A$ the rotation matrix about

$\frac{1}{3}(1, 2, 2)^T$.

Thm $\implies$ matrix representing $T_A$ wrt to $B$ is $R_\theta = M^{-1}$

Hence $A =$

as before.

**Eigenvectors**

From now on study lin maps of form $T : V \rightarrow V$
i.e. where domain =

**Philosophy of e-vectors** Linear $T : V \longrightarrow V$ often pick out their own preferred coord

Key to finding this preferred

**Defn 1** Let $T : V \longrightarrow V$ be

$v \in V - 0$, $\lambda$ a

s.t. $Tv$

We call $\lambda$ an

& $v$ an eigenvector with

As usual, for $A \in M_{nn}(\mathbb{F})$ an e-vector or

**e.g. 2 again** $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ rotn about line $\text{Span}(f)$. 
e.g. \( T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \) reflection about line \( \text{Span} \ f_1 \)

\( f_1 \) is an e-

since

Let \( f_2 \) be normal to

N.B. Basis \( \{ f_1, f_2 \} \) gives natural coord for

e.g. \( D \) Diagonal matrices

\( D = \)
Then $D e_i =$

so $e_i$ is an

N.B. Standard basis is a good basis in this case ∴ fn

$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = D$

$\implies y_i$ is a fn of

e.g. 5 $V = C^\infty$ is vect space of

Let $T : V \longrightarrow V$ be differentiation.

Then

Eigenbases

The preferred basis is given in

Defn 2 An eigenbasis for a linear transfor-
mation $T : V \rightarrow V$ is

We sim define the eigenbasis for a square e.g. 4 again The standard basis is an eigenbasis

**Thm 2** Let $T : V \rightarrow V$

& $B = \{v_1, \ldots\}$

with corresponding e-values $\lambda_1, \ldots$

The matrix $D$ representing $T$

Proof: Gen matrix reprn thm lect 16 $\implies$

$D = ([T$
e.g. 6 Find the matrix $A$ with e-values 1,2 and corresponding e-vectors

$f_1 =$

Ans: Let $M = (f_1 f_2) =$

Matrix representing $A$ wrt $\{f_1, f_2\}$ is