

Lecture 18: Solving linear equations. Image

Aim Lecture For $T : V \longrightarrow W$ linear,
understand when you can solve

Image

Defn-Prop Let $T : V \longrightarrow W$ be linear.

The image of T ,

$\text{im } T :=$

is a

We define $\text{rank } T :=$

Proof: a) $\text{im } T \ni T\mathbf{0}$

b) For $\mathbf{w}, \mathbf{w}' \in \text{im } T$, there are \mathbf{v}, \mathbf{v}'

with

Then $\mathbf{w} + \mathbf{w}' =$

so $\text{im } T$ is closed

c) If $\lambda \in \mathbb{F}$ & $\mathbf{w} \in T$

$$\lambda \mathbf{w} =$$

so $\text{im } T$ is closed under scalar multn too.

$$\therefore \text{im } T \leq$$

Rem $\text{im } T$ consists of vectors \mathbf{b} for

e.g. 1 Let $A \in M_{mn}(\mathbb{F})$ & T_A be

Define $\text{im } A =$

$$= \{T_A \mathbf{v} =$$

$$= \text{col}($$

Define also $\text{rank } A =$

sub-e.g.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

$A \longrightarrow$

$\therefore \text{im } A$ is spanned by

& rank $A =$

e.g. 2 $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ orthog projn onto

Computing image

Can often use general matrix reprn thm &

Prop 1 Let $T : V \longrightarrow W$ be

& B_V, B_W be finite ordered

Let A be the matrix

Then $[\text{im } T]_{B_W} =$

Why? Recall for $\mathbf{v} \in V, [T \mathbf{v}$

As \mathbf{v} varies over $V,$

LHS ranges over elts

RHS

e.g. 3 Let $M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Define $T : M_{22}(\mathbb{F}) \longrightarrow M_{22}(\mathbb{F})$ by $TC := MC$. Note that T is linear by

Find a basis for $\text{im } T$ and $\text{rank } T$.

Ans: First find matrix $A \in M_{44}(\mathbb{F})$ representing T wrt

$$\mathbf{e}_{11} =$$

$$T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$T \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$T \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $A = ([T$

=

Compute $\text{col}(A) = \text{im } A$.

$$A \longrightarrow$$

Hence a basis is

$$\mathbf{v}_1 =$$

They correspond to

$$C_1 =$$

Prop 1 & lemma 2 of lect 13,

$\{C_1, C_2\}$ is

rank

Rank-Nullity theorem

Thm 1 (Matrix version)

Let $A \in M_{mn}(\mathbb{F})$ & U be a

a) null $T =$ no. non-leading columns

b) rank $T =$

c) (Rank-Nullity)

no. col

Proof:

a) Each non-leading column in U gives
a param in

\therefore gives a basis

\therefore null $A =$

& a) holds.

b) Thm 1, lecture 13 gives basis

$\{\mathbf{v}_i \mid i$

so b) holds.

c) Add a) & b)

Thm 2 Let $T : V \longrightarrow W$ be a linear map,
 V, W finite dim.

Then $\dim V =$

Why? Choosing bases for V, W we can represent

$\text{null } A =$

$\text{rank } A =$

$\dim V =$ no. vectors in basis for

$=$

so thm 1 \implies

e.g. 4 A geometric picture illustrating rank-nullity thm

Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be an orthog projn onto 1-dim subspace W .

Applications of the rank-nullity theorem

Defn A hyperplane in \mathbb{R}^n is the set of solns to an eqn of form

$$a_1x_1 +$$

for some $a_i \in \mathbb{R}$ not all

e.g. 5 We'll use the rank-nullity thm to show that the intersection of 2 hyperplanes through $\mathbf{0}$ in \mathbb{R}^n has

A Consider hyperplanes

$$H_a : a_1x_1 +$$

$$H_b : b_1$$

$H_a \cap H_b$ is the set of simultaneous solns

i.e. this is ker

$$A =$$

$$\therefore \dim H_a \cap H_b =$$

$$\text{But } \dim A \leq \mathbb{R}^2 \implies \text{rank}$$

$$\text{So } \dim A = n -$$

ex Can you generalise this to m hyperplanes?

Prop 2 Let $T : V \longrightarrow W$ be linear map between finite dim vect spaces.

a) If T is invertible then $\dim V =$

b) Suppose $\dim V = \dim W$. Then T is invertible if either

Proof: of a) only. b) sim.

Can use defn to see T takes bases to bases or appeal to rank-nullity thm as follows.

$$T \text{ 1-to-1 } \implies \ker$$

T onto \implies im

Thm 2 \implies $\dim V = \text{null } T +$