# Lecture 17: Kernels. Linear Equations in Abstract Vector Spaces

**Aim Lecture** Understand in what sense an eqn such as $\frac{dy}{dx} + x^2 y =$

Study homogeneous

## Kernels

**Defn** Let $T : V \longrightarrow W$ be linear. The kernel of $T$ is

$\ker T :=$

i.e. "Homogeneous solns". See in prop 1 below that kernels are subspaces so can define the nullity of $T$ to be $\operatorname{null} T :=$

If $A \in M_{mn}(\mathbb{F})$ then define $\ker A =$

i.e. solns to

**e.g. 1** $T = \frac{d}{dx} : \mathbb{P} \longrightarrow \mathbb{P}$ has kernel polynomial solns to

$\ker T =$

$\operatorname{null} T = \dim_{\mathbb{R}}$

## Kernels are subspaces

**Propn 1** If $T : V \longrightarrow W$ is linear then $\ker T$ is a

Proof: a) Prop 1 lect 15 shows $T\mathbf{0} =$

b) If $\mathbf{v}, \mathbf{v}' \in \ker T$ then

c) If $\lambda \in \mathbb{F}$ then

Hence $\ker T$

**Dumb e.g. 2** Show

$W = \{\mathbf{x} \in \mathbb{R}^4 \,|x_1+$

is a subspace of $\mathbb{R}^4$.

Ans:

## Link with linear ODEs

**e.g. 3** Multn by a fn is linear.

For $g(x) \in \mathcal{R}[\mathbb{R}]$, define $T : \mathcal{R}[\mathbb{R}] \longrightarrow \mathcal{R}[\mathbb{R}]$ by

Then $T$ is lin by the

distrib law: $g(x)(f_1$

& comm law:

**e.g. 4** Define $T : \mathbb{P}_2(\mathbb{R}) \longrightarrow \mathbb{P}_3(\mathbb{R})$ by

$$Tp = (1+x)\frac{d^2p}{dx^2} - \frac{dp}{dx}$$

Note $T$ is lin being a lin combn of the lin maps

$\frac{d}{dx} : p(x) \mapsto$

&

The latter is linear being

**Q** Find the matrix reprn of $T$ wrt bases

$B_2 = \{$

$B_3 = \{$

Ans: $A = ([T1]_{B_3}$

$T1 =$

$[T1]_{B_3} =$

$Tx =$

$[Tx]_{B_3} =$

$Tx^2 =$

$[Tx^2]_{B_3} =$

Hence, $A =$

## Computing Kernels

Often can use matrix reprn thm &

**Prop 2** Let $T : V \longrightarrow W$ be

& $B_V, B_W$ be finite ordered

Let $A$ be the matrix

Then $[\ker T]_{B_V} =$

Why? Recall $[T\,\mathbf{v}]_{B_W} =$

so $T\,\mathbf{v} = \mathbf{0}$ iff

**e.g. 4 revisited** What's $\ker T$?

**A** $\ker A$ consists of vectors

$\mathbf{v} =$

$=$

Basis for $\ker A =$

Prop 2 $\implies$ the $p(x) \in \ker T$ are those of form

Lemma 2 lect 13 $\implies$ a basis for $\ker T$ is

Hence $\operatorname{null} T =$

& $\ker T =$

Check: $T1 =$

$T(x^2+$

## Inhomogeneous equations

The nature of solns to $T\mathbf{v} = \mathbf{w}$ is sim to $\mathbb{F}^m$ case as following prop shows.

**Prop 3** Let $T : V \longrightarrow W$ be linear. Suppose given $\mathbf{w} \in W$ and a particular soln

$\mathbf{v} = \mathbf{v}_p$ to eqn

(*)

The complete set of

Proof: As in session 1. Don't believe me? Observe

$\mathbf{v}_h + \mathbf{v}_p$ is a soln since

If $\mathbf{v}$ is a soln so $T\,\mathbf{v}$

then

$\implies \mathbf{v}_h := \mathbf{v} - \mathbf{v}_p \in$

Hence $\mathbf{v}$

**Cor** If $\ker T = \mathbf{0}$ then $T$ is

Why? If $\ker T$

## Geometric example

**E.g. 6** Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be orthogonal projn onto 1-dim subspace $L$