Lecture 17: Kernels. Linear Equations in Abstract Vector Spaces

**Aim Lecture** Understand in what sense an eqn such as $\frac{dy}{dx} + x^2y = \ldots$

Study homogeneous Kernels

**Defn** Let $T : V \rightarrow W$ be linear. The kernel of $T$ is

$\ker T := \ldots$

i.e. “Homogeneous solns”. See in prop 1 below that kernels are subspaces so can define the nullity of $T$ to be $\null T := \ldots$

If $A \in M_{mn}(\mathbb{F})$ then define $\ker A = \ldots$

i.e. solns to
\textbf{e.g. 1} \( T = \frac{d}{dx} : \mathbb{P} \rightarrow \mathbb{P} \) has kernel polynomial solns to

\[ \ker T = \]

\[ \text{null } T = \dim_{\mathbb{R}} \]

\textbf{Kernels are subspaces}

\textbf{Propn 1} If \( T : V \rightarrow W \) is linear then \( \ker T \) is a

Proof: a) Prop 1 lect 15 shows \( T0 = \)

b) If \( \mathbf{v}, \mathbf{v}' \in \ker T \) then

c) If \( \lambda \in \mathbb{F} \) then
Hence \( \ker T \)

**Dumb e.g. 2** Show

\[ W = \{ \mathbf{x} \in \mathbb{R}^4 | x_1 + \] is a subspace of \( \mathbb{R}^4 \).

**Ans:**

**Link with linear ODEs**

**e.g. 3** Multn by a fn is linear.

For \( g(x) \in \mathcal{R} [\mathbb{R}] \), define \( T : \mathcal{R} [\mathbb{R}] \longrightarrow \mathcal{R} [\mathbb{R}] \) by

Then \( T \) is lin by the
distrib law: \( g(x)(f_1) \)

& comm law:
**e.g.** 4 Define $T : P_2(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ by

$$T p = (1 + x) \frac{d^2 p}{dx^2} - \frac{dp}{dx}$$

Note $T$ is lin being a lin combn of the lin maps

$$\frac{d}{dx} : p(x) \mapsto$$

&

The latter is linear being

**Q** Find the matrix reprn of $T$ wrt bases

$B_2 = \{ \}$

$B_3 = \{ \}$

Ans: $A = ([T1]_{B_3}$

$T1 =$
\[ [T1]_{B3} = \]

\[ Tx = \]

\[ [Tx]_{B3} = \]

\[ Tx^2 = \]

\[ [Tx^2]_{B3} = \]

Hence, \( A = \)

Computing Kernels

Often can use matrix reprn thm &

**Prop 2** Let \( T : V \rightarrow W \) be
& \ B_V, B_W \ be \ finite \ ordered

Let \ A \ be \ the \ matrix

Then \ [\ker T]_{B_V} = \]

Why? Recall \ [Tv]_{B_W} = \]

so \ Tv = 0 \ iff \]

\textbf{e.g. 4 revisited} \ What’s \ ker T? \]

\textbf{A ker A consists of vectors} \]

\ v = \]

= \]

Basis for ker A = \]

6
Prop 2 $\implies$ the $p(x) \in \ker T$ are those of form

Lemma 2 lect 13 $\implies$ a basis for $\ker T$ is

Hence $\text{null } T =$

& $\ker T =$

Check: $T1 =$

$T(x^2+$

**Inhomogeneous equations**

The nature of solns to $T\mathbf{v} = \mathbf{w}$ is sim to $\mathbb{F}^m$ case as following prop shows.

**Prop 3** Let $T : V \longrightarrow W$ be linear. Suppose given $\mathbf{w} \in W$ and a particular soln
\( \mathbf{v} = \mathbf{v}_p \) to eqn

(\ast)

The complete set of

Proof: As in session 1. Don’t believe me?

Observe

\( \mathbf{v}_h + \mathbf{v}_p \) is a soln since

If \( \mathbf{v} \) is a soln so \( T \mathbf{v} \) then

\[ \implies \mathbf{v}_h := \mathbf{v} - \mathbf{v}_p \in \]
Hence $v$

**Cor** If $\ker T = 0$ then $T$ is

Why? If $\ker T$

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**Geometric example**

**E.g. 6** Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be orthogonal projn onto 1-dim subspace $L$