Lecture 16: Algebra of linear transformations

Aim Lecture See how the algebra of linear maps mimics the algebra

Linear structure

Defn 1 Let $T, T' : V \longrightarrow W$ be

Let $\lambda, \mu \in \mathbb{F}$. Define new maps

Sum: $T + T' : V \longrightarrow W$ by

scalar multiple: $\lambda T : V \longrightarrow W$ by

Linear Combn: $\lambda T + \mu T' : V \longrightarrow W$ by

Propn 1 $T + T', \lambda T$ are linear too. Hence so is $\lambda T + \mu T'$.

Proof: Check λT lin: for $\mathbf{v}, \mathbf{w} \in V, \alpha \in \mathbb{F}$

Add Condn: $(\lambda T)(\mathbf{v} + \mathbf{w}) \stackrel{\text{def}}{=}$

$= \lambda(T\,\mathbf{v} +$

$= \lambda(T\,\mathbf{v})$

$=$

so addn condn holds.

Scalar Multn Condn: $(\lambda T)(\alpha\,\mathbf{v}) =$

So scalar multn condn also holds &

Check $T + T'$ linear: Can prove as above.

Here we only prove in case $V = \mathbb{F}^n, W = \mathbb{F}^m$ so by matrix representation thm there are $A, B \in M_{mn}(\mathbb{F})$ with

$(T + T')\,\mathbf{v} =$

$T + T'$ is matrix

Proof gives

Formula 1 $T_A + T_B = T_{A+B}$

Sim $\lambda T_A = T_{\lambda A}$.

Fact: Let V, W be vect spaces / $\mathbb{F}$ &

$\text{Hom}_{\mathbb{F}}(V, W)$ be the set of

Then sum & scalar multiple above are an addn law &

making $\text{Hom}_{\mathbb{F}}(V, W)$ a

No Proof: But note the matrix reprn thm means $\text{Hom}_{\mathbb{F}}(\mathbb{F}^n, \mathbb{F}^m)$ is essentially

Application to orthogonal projections

e.g. 1 Identity linear map

For vect space V, let $\mathrm{id} : V \longrightarrow V$ be the function $\mathrm{id}\,\mathbf{v} = \mathbf{v}$. It is linear (CHECK!).

$\mathrm{id} : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ is represented by

since $\mathrm{id}\,\mathbf{v} = \mathbf{v} =$

e.g. 2 Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ be an o/n

Let $P_i :=$ projn

What's $P_{12} :=$ projn onto

Method 1: $P_{12}\,\mathbf{v} = \mathrm{id}\,\mathbf{v} -$

$\therefore P_{12} = \mathrm{id}$

so is linear being

Method 2: also $P_{12}\,\mathbf{v} =$

$\therefore P_{12}$ is matrix multn by

I_3-

Composing linear maps

Propn 2 Let $T : V \longrightarrow W, S : U \longrightarrow V$ be linear maps. Then $T \circ S :$

Proof: ex in checking axioms. Here do case $U = \mathbb{F}^n, V = \mathbb{F}^m, W = \mathbb{F}^l$. Matrix rep thm $\Longrightarrow$

For some $A \in$

For some $B \in$

$(T \circ S)\,\mathbf{v} =$

Formula 2

Example of rotations

Let $T_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be anti-clockwise

What's $T\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$?

Use complex numbers.

$e^{i\theta}(x + iy) =$

$$\therefore T \begin{pmatrix} x \\ y \end{pmatrix} =$$

Hence $T_\theta = T_{R_\theta}$ is linear. R_θ is

e.g. 3 Let's verify directly formula 2 for composing T_θ & T_ϕ.

Geom considerations $\implies T_\theta \circ$

$R_\theta R_\phi =$

$=$

So $T_{R_\theta R_\phi} =$

Invertible linear maps

Recall following defns from calculus regarding a function $f : X \longrightarrow Y$.

Defn 2 1) We say that f is one-to-one (1-1) or injective if for any $y \in Y$, the soln

i.e. $f(x) = f(x')$

2) $f : X \longrightarrow Y$ is onto or

i.e. $\mathrm{im} f$

Recall also from calculus

Facts a) A fn $f : X \longrightarrow Y$ is invertible iff f is 1-1 &

b) In this case, the eqn $f(x) = y$ always has a

denoted $x =$

c) $f \circ f^{-1} =$

e.g. 4 For a basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of vect space V over $\mathbb{F}$, $S_B : V \longrightarrow \mathbb{F}^n$ defined by

Why? Given any $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbb{F}^n$, the unique soln

Propn 3 If $T : V \longrightarrow W$ is linear and invertible then

Proof: in case $T = T_A$ for $A \in M_{mn}(\mathbb{F})$.
Since T is invertible, $T\,\mathbf{x} = A\,\mathbf{x} = \mathbf{b}$ always

Hence A is

$\therefore T^{-1} =$

Formula 3 T_A

e.g. 5 Geometrically see "inverse" rotation

$T_\theta^{-1} =$

Corresponds on matrix side to

$R_\theta^{-1} =$

$=$

Generalised matrix representation theorem

Finite bases allow us to represent vect spaces by $\mathbb{F}^n$ & lin maps between vector spaces by

Thm (Generalised Matrix Rep Thm)

Let $T : V \longrightarrow W$ be

$B_V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an

$B_W = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be an

Let $A = (\mathbf{a}_1 \dots \mathbf{a}_n)$ be

with columns

Then $(*)$ $\qquad [T\,\mathbf{v}]_{B_W} =$

Why? Both sides of (*) are linear in $\mathbf{v}$ since they are composites

$\therefore$ suffice check (*) holds for $\mathbf{v} =$

LHS $= [T\,\mathbf{v}_i]_{B_W} =$

RHS $=$

e.g. 6 Rotation matrix via reprn thm

Let $V =$ vect space of geom vectors in 2-dim

Let $T : V \longrightarrow V$ be lin map which rotates

We put coords on V by letting $\mathbf{u}_1$ be any length

Rotate $\mathbf{u}_1$ 90°

Then $B = \{\mathbf{u}_1,$

Q What matrix R represents

A Trig $\implies$ $T\,\mathbf{u}_1 =$

so $[T\,\mathbf{u}_1$

Sim $[T$

Thm $\implies T$ is

wrt basis

$T_R : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ captures all the