

## Lecture 14: Some bases for polynomials

**Aim Lecture**  $\{1, x, \dots, x^d\}$  is not always the most useful basis for  $\mathbb{P}_d$ . We

### Motivational Example

e.g. **1** Lex believes

Experiment:

	1	2	3	4
Strength	9.7	9.2	7.9	4.9

Typical questions:

Interpolation: estimate strength when

Extrapolation: estimate strength when

# Method

## Polynomial interpolation

Consider data points

$(t_0, y_0), \dots, (t_n, y_n)$  where

Seek polynomial

$$y(t) = \lambda_0 +$$

s.t.

**Notn:**  $\mathbf{t} =$

$$y(\mathbf{t}) =$$

Want  $y(\mathbf{t}) = \mathbf{y}$  i.e. if

$$A := \begin{pmatrix} 1 & t_0 & t_0^2 & \dots & t_0^n \\ 1 & & & & \\ \vdots & & & & \\ 1 & & & & \end{pmatrix}$$

then

**Thm 1** There's a unique poly

$$y(t) = \lambda_0 +$$

s.t.  $y(t_i) = y_i$ . Its coeff are given

Proof: Cor a) lect 4  $\implies$  if  $A\lambda = \mathbf{y}$  has  
solns then

$\therefore$  columns of

$\therefore$  row-echelon form has all

But  $A$  is square so it must be

**e.g. 1 cont'd** see MATLAB lexl.m

## Lagrange polynomials

Consider  $\mathbf{t} =$

with  $t_i$  distinct. For  $j = 0, \dots, n$  consider

Lagrange

$$P_j(t) := \prod_{k \neq j} \frac{t_j - t_k}{t_j - t_k}$$

**e.g. 2** If  $\mathbf{t} = (1, 2, 3)$  then

$$P_0(t) =$$

$$P_1(t) =$$

$$P_2(t) =$$

**Thm 2** a)  $P_j(t_j) =$

b)  $P_j(t$

c)  $B = \{P_0(t), \dots,$

d) If  $y(t)$  is poly with  $y(\mathbf{t}) = \mathbf{y}$  then

(\*)  $y(t) = y_0 P$

i.e.  $[y(t)]_B =$

Proof: a) & b) follow

c)  $\dim \mathbb{P}_n = n + 1$  so suffice

Suppose  $\lambda_0 P_0(t) +$

For any  $i =$

$0 = \lambda_0 P$

$=$

$\therefore P_0(t), \dots$

d) Just note

$y_0 P_0(t_i) +$

so both sides of (\*) have the same

**e.g. 1 again** If  $\mathbf{t} = (1, 2, 3, 4)^T$  then desired

$$y(t) = 9.7P$$

### Numeric polynomials

**Defn 1** A poly  $p(t)$  with coeff in  $\mathbb{Q}$  is numeric if

**e.g. 3**  $t^2 -$

$$\frac{1}{2}(t^2$$

**Defn 2** For any  $\mathbb{R}$ -valued fn  $p(t)$  we define the difference

$$\Delta p(t) = p(t +$$

**e.g. 4**  $\Delta(t^3) =$

**e.g. 5** For  $k \in \mathbb{N}$ , define

$$\binom{t}{k} :=$$

Then  $\Delta$

**Lemma** Suppose  $f(t), g(t) \in \mathbb{P}$  with

$$\Delta f(t) =$$

Proof: Let  $c = f(0) -$

We show by induction that  $f(n) = g(n)$  for any  $n \in \mathbb{N}$ . Note

$$f(n+1) - f(n) = (\Delta f)(n) = g(n)$$

$$\text{So } f(n+1) - g(n+1) = f(n) - g(n)$$

Hence  $f(t) = g(t) + c$

**Thm 3** a)  $B = \{1 =$

b) If  $p(t)$  is numeric then  $[p(t)]_B$  has

Proof: a) The coords of

Columns lin indep  $\implies B$

$\therefore B$  is a basis as  $\dim \mathbb{P}_n$

b) Use induction on deg

If  $p(t)$  is constant

Otherwise note  $\Delta p(t)$  is numeric

Inductive hyp  $\implies$

$$\Delta p(t) = c_1$$

for some

Let  $q(t) = c_1$

so that  $\Delta q(t) =$

Lemma  $\implies$

Hence  $p(t) =$

e.g. 5 suggests

**Formula**  $\binom{0}{k} + \binom{1}{k} + \dots + \binom{n}{k}$

Proof: Induction on  $n$ .  $n = 0$  holds as

For  $n > 0$

**e.g. 6** Find  $1^2 + 2^2 + \dots + n^2$

**A** Write  $t^2$  in terms of

$$\therefore t^2 =$$
$$\sum_{t=0}^n t^2 =$$