Lecture 13: Constructing Bases

**Aim Lecture** Give effective means of

**Bases for** $\text{Span}(S) \subseteq \mathbb{F}^m$

Key: A basis is a minimal

Can reduce spanning set to a

**Thm 1** Let $A = (v_1 \ v_2 \ldots v_n)$. Recall

$\text{col}(A) = \ldots$

Let $U$ be a

Then we have the following basis of $\text{col}(A)$

$B = \{v_i\}$

Proof: will be clear from following e.g. (also see notes §7.7.3 thm 6)

**e.g. 1** Find a basis for $\text{col}(A)$ where
$$A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & -3 & 2 & 1 \end{pmatrix} = (\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \mathbf{v}_4)$$

1st & 2nd

Thm 1 $\implies$
Why did it work?

i.e. why’s \{ \mathbf{v}_1 \}

Check lin indep: Omitting 3rd & 4th column

from above calculation, we see

\[ \therefore \text{only soln to } x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 = 0 \text{ is} \]

\[ \therefore \mathbf{v}_1, \mathbf{v}_2 \text{ are} \]

Check span: 3rd & 4th columns correspond to parameters in soln

Pick soln \( \mathbf{x} \) with

Back substn \( \implies \)

\[ 0 = A \mathbf{x} = \]
Sim, setting
get soln \( x = \)

so \( v_4 \in \)
\[ \therefore \text{Span}(v_1, v_2) = \]

Hence, \( \{v_1, v_2\} \) is lin indep & spans \( \text{col}(A) \)
so is a basis.

**Extending linearly independent sets in** \( \mathbb{F}^m \)

Max

**Thm 2** Let \( W \) be a subspace of \( \mathbb{F}^m \) and
\( S = \{v_1, \ldots, v_n\} \subset W \)
Suppose \( \{w_1, \ldots, w_r\} \)
Then applying method of thm 1 to
\{v_1,\}

Note: 1) We can apply thm 1 since
\{v_1,\}
2) \{v_1, \ldots, v_n\} lin indep \implies

\therefore the basis \textbf{B} produced by this method

Hopefully, the reason why this works will be clear from the following e.g.

\textbf{e.g. 2} Let \( S = \{v_1 = (1, 2, -1)^T, v_2 = (3, 2, -1)^T\} \). Extend \( S \) to a basis of \( \mathbb{R}^3 \).

Ans: Note vectors not parallel \implies
Let \( w_1, \)

Let \( w_1, \)
cont’d

1st, 2nd & 4th

i.e.

**Remark** What happens to this method if $S$ is lin dependent so that $S$ cannot form part of a basis?

Ans: Method produces a basis with as many members of $S$ as possible.

**Bases for subspaces defined by equations**

e.g. 3 Let
\[ A = \begin{pmatrix} 1 & 0 & 3 & 1 & -1 \\ 0 & 1 & 2 & -1 & 4 \end{pmatrix}. \]

You can check \( W := \{ \mathbf{x} \in \mathbb{R}^5 \mid A \mathbf{x} \} \)

What’s a basis?

A General soln

\[ \mathbf{x} = \]

Thus a basis for \( W \) is
Subspaces of $M_{mn}(\mathbb{F})$ and $\mathbb{P}_d$

We reduce to the $\mathbb{F}^n$ case via coordinates.

**Lemma 1** Let $V = \text{vector space/ field } \mathbb{F}$

Let $B$ be a basis with $n$ elements.

Let $W \subseteq V$

Define $[W]_B = \{ \}

i.e. set of coords

If $W \leq V$, then $[W]_B \leq$

Proof: Just check closure axioms. e.g. for $w_1, w_2 \in$

closure under addn given by

**e.g. 4** Let $V = \mathbb{P}_2$ & $B = \{1, x, \}$

Let $W = \text{Span}(1, x) = \{ \}$
$[W]_B = \{ [\]

= \text{Span}($

**Lemma 2** Let $V, B, W$ be as in Lemma 1. Let $S = \{ w_1, \ldots, w_m \} \subset W$. Then

a) $S$ is lin indep iff

b) $S$ spans $W$ iff

Proof: a) is just scholium lect 11.

$$\text{Span}([S]_B) =$$

$$\{ \lambda_1 w_1$$

$$\lambda_1 w_1$$

$$= [\text{Span}(S)]_B.$$ Thus if $S$ spans $W$ then
Conversely, if \( [\mathbf{w}_1]_B, \ldots, [\mathbf{w}_m]_B \) then \( [\text{Span}(S)]_B = \)
i.e. \( \text{Span}(S) = \)

**e.g. 5** From e.g.1 & lemma 2 we see basis for \( \text{Span}(1+x+x^2, -1+x-3x^2, 1+2x^2, 2+3x+x^2) \)

### Bases in other vector spaces

\( \mathbb{R}^\infty := \{ \)

is a vect space / \( \mathbb{R} \) with

addn: \( (x_i) + (x'_i) = \)

scalar multn:

Solns to dts can be considered elts of
e.g. 6 Let $W = \{(x_i) \in \mathbb{R}^\infty |$
\[ x_n - 2x_{n-1} + x_{n-2} = 0, \quad n \geq 2 \}$
Can show $W \leq \mathbb{R}^\infty$. Find a basis.

A Can use lect 6 material on dts OR
Note $(x_i) \in$
\[ d := x_n - x_{n-1} = \]
i.e. $(x_0, x_1, \ldots)$ is an
with initial
\[ \therefore (x_i) = (x_0, x_0 + \]
\[ = x_0(1, \]