

## Lecture 13: Constructing Bases

**Aim Lecture** Give effective means of

Bases for  $\text{Span}(S) \subseteq \mathbb{F}^m$

Key: A basis is a minimal

Can reduce spanning set to a

**Thm 1** Let  $A = (\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n)$ . Recall

$\text{col}(A) =$

Let  $U$  be a

Then we have the following basis of  $\text{col}(A)$

$B = \{\mathbf{v}_i$

Proof: will be clear from following e.g. (also see notes §7.7.3 thm 6)

**e.g. 1** Find a basis for  $\text{col}(A)$  where

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & -3 & 2 & 1 \end{pmatrix} = (\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \mathbf{v}_4)$$

**A**

1st & 2nd

Thm 1  $\implies$

## Why did it work?

i.e. why's  $\{\mathbf{v}_1$

Check lin indep: Omitting 3rd & 4th column  
from above calculation, we see

$\therefore$  only soln to  $x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 = \mathbf{0}$  is

$\therefore \mathbf{v}_1, \mathbf{v}_2$  are

Check span: 3rd & 4th columns correspond  
to parameters in soln

Pick soln  $\mathbf{x}$  with

Back substn  $\implies$

$$\mathbf{0} = A \mathbf{x} =$$

Sim, setting

get soln  $\mathbf{x} =$

so  $\mathbf{v}_4 \in$

$\therefore \text{Span}(\mathbf{v}_1, \mathbf{v}_2) =$

Hence,  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is lin indep & spans  $\text{col}(A)$   
so is a basis.

Extending linearly independent sets in  $\mathbb{F}^m$

Max

**Thm 2** Let  $W$  be a subspace of  $\mathbb{F}^m$  and  
 $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset W$

Suppose  $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$

Then applying method of thm 1 to

$\{\mathbf{v}_1,$

Note: 1) We can apply thm 1 since

$\{\mathbf{v}_1,$

2)  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  lin indep  $\implies$

$\therefore$  the basis  $B$  produced by this method

Hopefully, the reason why this works will be clear from the following e.g.

**e.g. 2** Let  $S = \{\mathbf{v}_1 = (1, 2, -1)^T, \mathbf{v}_2 = (3, 2, -1)^T\}$ . Extend  $S$  to a basis of  $\mathbb{R}^3$ .

Ans: Note vectors not parallel  $\implies$

Let  $\mathbf{w}_1,$

cont'd

1st, 2nd & 4th

i.e.

**Remark** What happens to this method if  $S$  is lin dependent so that  $S$  cannot form part of a basis?

Ans: Method produces a basis with as many members of  $S$  as possible.

**Bases for subspaces defined by equations**

**e.g. 3** Let

$$A = \begin{pmatrix} 1 & 0 & 3 & 1 & -1 \\ 0 & 1 & 2 & -1 & 4 \end{pmatrix}.$$

You can check  $W := \{\mathbf{x} \in \mathbb{R}^5 \mid A\mathbf{x}$

What's a basis?

**A** General soln

$\mathbf{x} =$

$=$

Thus a basis for  $W$  is

## Subspaces of $M_{mn}(\mathbb{F})$ and $\mathbb{P}_d$

We reduce to the  $\mathbb{F}^n$  case via coordinates.

**Lemma 1** Let  $V =$  vector space/ field  $\mathbb{F}$

Let  $B$  be a basis with  $n$  elements.

Let  $W \subseteq$

Define  $[W]_B = \{$

i.e. set of coords

If  $W \leq V$ , then  $[W]_B \leq$

Proof: Just check closure axioms. e.g. for

$\mathbf{w}_1, \mathbf{w}_2 \in$

closure under addn given by

**e.g. 4** Let  $V = \mathbb{P}_2$  &  $B = \{1, x,$

Let  $W = \text{Span}(1, x) = \{$

$$[W]_B = \{[w_1]_B, \dots, [w_m]_B\}$$

$$= \text{Span}([w_1]_B, \dots, [w_m]_B)$$

**Lemma 2** Let  $V, B, W$  be as in Lemma 1.

Let  $S = \{\mathbf{w}_1, \dots, \mathbf{w}_m\} \subset W$ . Then

- a)  $S$  is lin indep iff
- b)  $S$  spans  $W$  iff

Proof: a) is just scholium lect 11.

$$\text{Span}([S]_B) =$$

$$\{\lambda_1[\mathbf{w}_1]_B, \dots, \lambda_m[\mathbf{w}_m]_B\}$$

$$\lambda_1 \mathbf{w}_1, \dots, \lambda_m \mathbf{w}_m$$

$$= [\text{Span}(S)]_B.$$

Thus if  $S$  spans  $W$  then

Conversely, if  $[\mathbf{w}_1]_B, \dots, [\mathbf{w}_m]_B$

then  $[\text{Span}(S)]_B =$

i.e.  $\text{Span}(S) =$

**e.g. 5** From e.g.1 & lemma 2 we see basis  
for  $\text{Span}(1 + x + x^2, -1 + x - 3x^2, 1 + 2x^2, 2 + 3x + x^2)$

## Bases in other vector spaces

$\mathbb{R}^\infty := \{$

is a vect space /  $\mathbb{R}$  with

addn:  $(x_i) + (x'_i) =$

scalar multn:

Solns to dts can be considered elts of

**e.g. 6** Let  $W = \{(x_i) \in \mathbb{R}^\infty \mid$   
 $x_n - 2x_{n-1} + x_{n-2} = 0, n \geq 2\}$

Can show  $W \leq \mathbb{R}^\infty$ . Find a basis.

**A** Can use lect 6 material on dts OR

Note  $(x_i) \in$

$$d := x_n - x_{n-1} =$$

i.e.  $(x_0, x_1, \dots)$  is an

with initial

$$\therefore (x_i) = (x_0, x_0 +$$

$$= x_0(1,$$