Lecture 11: Bases

Aim Lecture Study coord

Study the function Span

Basis

Defn 1 Let V = vect space / field $\mathbb{F}$. $B \subset V$ is a

1) Span(B) =

& 2) B is lin

Equiv, B is a basis if any $\mathbf{v} \in V$ can be written

e.g. 1 $V = \mathbf{0}$ has basis

e.g. 2 $\mathbb{P}$ has basis $B =$

since B spans $\mathbb{P}$

& $0 = \lambda_0 +$

so B is also

e.g. 3 $V = M_{mn}(\mathbb{F})$. For $i \in$

define $E_{ij} =$

i.e. 0 everywhere but

$B = \{E_{ij}\}$ is a basis. e.g. for M_{22} we can uniquely express

Linearity of coordinates

Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an ordered basis for vect space V / $\mathbb{F}$.

Rem Then any $\mathbf{v} \in V$ has coord

where, $\mathbf{v} = x_1\,\mathbf{v}_1 +$

Thm 1 For $\mathbf{u}, \mathbf{w}, \in V$, $\alpha \in \mathbb{F}$:

a. $[\mathbf{u} + \mathbf{w}]_B =$

b.

N.B. 1. This means fn $V \longrightarrow \mathbb{F}^n : \mathbf{u} \mapsto [\mathbf{u}]_B$ is

2. Coord allow you to "identify" V with $\mathbb{F}^n$ the same way it allows you to "identify" 3-dim space with

Proof: Suppose $\mathbf{u} = u_1\,\mathbf{v}_1 +$

$\mathbf{w} =$

a) $[\mathbf{u} + \mathbf{w}]_B =$

$= (u_1 + w_1,$

$= (u_1,$

$=$

b) Sim.

e.g. 4 $B = \{e^x, e^{2x}, e^{3x}\} \subset C(\mathbb{R})$ is a basis for Span(B). Show $f_1 = e^x, f_2 = e^x + e^{2x}, f_3 = e^x + e^{2x} + e^{3x}$ is

A If $\lambda_1 f_1 +$

then $0 = [\lambda_1 f_1 +$

$= [\lambda_1 f_1]_B +$

$= \lambda_1$

But $[f_1]_B =$ $[f_2]_B =$

$[f_3]_B =$

so $0 = \lambda_1 =$

$\therefore f_1, f_2, f_3$ are

Scholium The above argument shows that $\mathbf{w}_1, \ldots, \mathbf{w}_m \in V$ are lin indep iff

Orthonormal bases

The most useful coord systems have orthogonal axes.

E.g. In $\mathbb{R}^3$

Recall $B \subset \mathbb{R}^m$ is orthonormal (o/n) if the vectors are

Propn 1 An o/n set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is lin indep

Proof: If

then for any i

Propn 2 If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an o/n basis of $\mathbb{R}^m$ (see later $m = n$) then

Proof: None. This is ex 60 of §7.8 of the notes.

Geometric picture is more enlightening.

Suppose $B = \{\mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^2$ is o/n.

Alternate characterisation of linear depend

Prop 3 If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is lin depend then there's some i s.t. $\mathbf{v}_i$

Conversely, if $\mathbf{v}_i \in$ $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n)$ then

Proof: See §7.6.4 thm 3. The proof there generalises

e.g. 5 Suppose non-trivial line reln holds

$2\,\mathbf{v}_1 + 2\,\mathbf{v}_3 -$

Then

Conversely, suppose $\mathbf{w}_1 = 3\,\mathbf{w}_2$

then

so $\{\mathbf{w}_1, \mathbf{w}_2$

How does Span(S) vary with S

Prop 4 V = vector space/ field $\mathbb{F}$

Let $S_1 \subseteq S_2$ be subsets of V. Then

Proof: Every element in Span(S_1)

is

so is

so is

i.e.

Thm 1 Let $S \subseteq V$ = vector space/ field $\mathbb{F}$

For $\mathbf{v} \in V$,

$$\text{Span}(S \cup \{\mathbf{v}\}) = \text{Span}(S)$$

iff $\mathbf{v} \in \text{Span}(S)$.

Proof: ($\implies$) If $\text{Span}(S \cup \{\mathbf{v}\}) = \text{Span}(S)$ then

($\Longleftarrow$) Suppose $\mathbf{v} \in \text{Span}(S)$. Recall that $\text{Span}(S \cup \{\mathbf{v}\})$ is the

But $\text{Span}(S)$ contains

Hence, $\text{Span}(S) \supseteq$

Prop 4 $\implies$ $\text{Span}(S) \subseteq$

The two inclusions show

N.B. Prop 3 $\implies$ if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ lin depend then some $\mathbf{v}_i \in \mathrm{Span}(S - \{\mathbf{v}_i\})$. So then

i.e. can omit $\mathbf{v}_i$ without

e.g. 6 $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ non-parallel.

Suppose $\mathbf{v}_3 \in W = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$