Lecture 10: Coordinates. Linear Dependence

Aim Lecture Generalise notion of parallel

Informal first look at coordinates

Consider non-parallel $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$

$W =$ the plane

Set up "grid" or "coordinate" system

$\mathbf{v}_1, \mathbf{v}_2$ determines coords on W

$\mathbf{v}_1$ has coords

N.B. Works $\because$ $\mathbf{v}_1, \mathbf{v}_2$

Can extend this to coord system on $\mathbb{R}^3$ by

provided $\mathbf{v}_3$

We study degenerate case where $\mathbf{v}_3 \in$

Linear dependence

Defn 1 Let $S \subseteq V =$ vect space / field $\mathbb{F}$.

S is linearly dependent if there are

distinct

scalars

with

i.e. some non-trivial linear combn of distinct vectors is zero.

e.g. 1 $V = \mathbb{R}^2 \ni (1,0), (0,1), (3,2)$

e.g. 2 $\mathbf{v}, \mathbf{w}$ are lin depend iff

Why? If $\lambda\,\mathbf{v} + \mu\,\mathbf{w} =$

either $\lambda \neq 0$ so $\mathbf{v} =$

Conversely, if $\mathbf{v} = \lambda\,\mathbf{w}$

so $\mathbf{v}, \mathbf{w}$

Defn 2 $S \subseteq V$ = vect space / field $\mathbb{F}$. Say S is linearly independent if it is not lin dependent.

i.e. If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ (with $\mathbf{v}_i$'s distinct) then the only soln to

Coordinates

Thm (Uniqueness of Linear Combn)

Let V = vect space / field $\mathbb{F}$ & S

$= \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ non-empty, lin

Any vector $\mathbf{v} \in \mathrm{Span}(S)$ can be written

$\mathbf{v} =$

i.e. If also $\mathbf{v} =$

then

Note: $\lambda_1, \ldots, \lambda_n$ give coordinates of $\mathbf{v}$.

Proof: Suppose

$\mathbf{v} =$

Subtracting using distributive, associative and commutative laws gives

Defn of lin indep $\implies$

Defn 3 If $B = \{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is an ordered lin

$\mathbf{v} \in \mathrm{Span}(B)$ then the coord

$[\mathbf{v}]_B :=$

where $\lambda_1, \ldots, \lambda_n$ are the

$\mathbf{v} =$

e.g. 3 $B = \{1, x, x^2\}$ is lin

$[a_0 + a_1 x + a_2 x^2]_B =$

Problem with lin depend

Defn 3 fails if B is

e.g. if $B = (1,0), (0,1), (3,2)$ then

Testing lin depend in $\mathbb{F}^m$

As in lecture 9, consider vectors in $\mathbb{F}^m$

$$\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \ldots, \mathbf{a}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Saw $x_1\,\mathbf{a}_1 + \ldots + x_n\,\mathbf{a}_n = A\,\mathbf{x}$ hence,

Propn $B = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is lin independent iff $A\,\mathbf{x} = \mathbf{0}$ has unique soln $\mathbf{x} = \mathbf{0}$. If $\mathbf{v} \in col(A)$ then $[\mathbf{v}]_B$ is soln

E.g. 2 $\mathbf{v}_1 = (1,1,2)^T, \mathbf{v}_2 = (2,1,1)^T, \mathbf{v}_3 = (1,2,5)^T, \mathbf{v}_4 = (0,0,1)^T$

Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ lin independent?

We solve $A\,\mathbf{x} = \mathbf{0}$.

cont'd

Third column not leading so

Propn $\Longrightarrow$

In fact $\mathbf{x} = (-3, 1, 1, 0)^T$ is a non-zero soln to $A\,\mathbf{x} = \mathbf{0}$ which corresponds to

the non-trivial relation

Also, above calculation shows $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ lin

Why? Omitting 3rd column from above calculation

all columns

Finding coordinates

e.g. 4 cont'd $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$. Let $\mathbf{v} = (1, 2, 1)$. Is $\mathbf{v} \in \text{Span}(B)$ and if so what is $[\mathbf{v}]_B$?

A Solve

$[\mathbf{v}]_B$

Working in other vector spaces

Reduce question to linear algebra question of the type in e.g. 2 as in the following

E.g. 5 Is $B = \{p_1(x) = 1+x+2x^2, p_2(x) = 2+x+x^2, p_3(x) = x^2\}$ lin independent in $\mathbb{P}$? If so what is coord of $1+2x+x^2$ wrt

A Try to solve

(*)

Equate coefficients:

This gives the same system of linear equations as in E.g. 4.