Lecture 1: Fields (a.k.a. Number systems).

Complex Numbers

**Aim Lecture** Extend the real “number

**Abstract Number Systems:** The number system $\mathbb{F} = \mathbb{Q}$

has addn &

which satisfy

For any $x, y, z$

1. Associative Law of Addition:

$$(x + y) + z =$$

2. Commutative Law of Addition:

$$x + y =$$
3. Existence of Zero: there exists an elt s.t. \( x + \)

4. Existence of Negatives: there exists an elt s.t. \( x + \)

5. Associative Law of Multn:
\[(xy)\]

6. Commutative Law of Multn:
\[xy = \]

7. Existence of One: there exists an elt s.t.

8. Existence of Inverses: If \( x \neq \)

9. Distributive Law:
\[ x(y + \ \ \ \ \ \ \ \ \ \ \ \]  

**Defn** Laws 1-9 are called

Let \( \mathbb{F} \) be any

a) an addn operation i.e. rule assigning

b) a multn operation

Say \( \mathbb{F} \) is a *field* if

**E.g. 1** \( \mathbb{F} = \mathbb{Q} \),

**Subtrn/Division** In any field, can define subtraction as adding the negative. Sim division is

Subtle Point [H, see §6.12]: Need uniqueness
of negatives, inverses, 0 and 1 to

**Complex numbers**

We remove deficiency of no real

**Thm** There’s a field $\mathbb{C}$ containing

a) You add & mult real

b) There’s an elt $i \in$

c) Any $z \in \mathbb{C}$ can be written in Cartesian

d) Let $a, a', b, b' \in \mathbb{R}$. If $a + bi = a'$


$a - a' =$
\[(a - a')^2 = (b- \]

\[\therefore a = a', b = b'.\]

**Defn** Elts \(a + bi \in \mathbb{C}\) are called

field axioms \(\implies\) manipulate complex

**Add** assoc & comm of addn \(\implies\)

\[(1 + 2i) + (3+ \]

\[= \]

**Subtrn** \(-(3 + 5i) = \]

so \((1 + 2i) - \]

**Multn** \((1 + 2i)(3 + 4i) = (1 + 2i \]

\[= 3+ \]
**Defn** Let \( z = a + bi \in \mathbb{C}, a, \)

Its complex conjugate is

**Formula** \( z\bar{z} = (a + bi)(a \)

**Division** \( \frac{1+2i}{3+4i} = \frac{1+2i}{3+4i} \)

\[
\frac{(1+\text{-})(3+\text{-})}{(3+\text{-})(3+\text{-})}
\]

Remark: Technically, we used formula

\((wz)^{-1} = \)

Holds \( \because wz \)

Properties of conjugation

**Formulae** For \( w, z \in \)
1. $\bar{z}$

2. $z - w$

4. $\overline{zw}$

5. $z$ is real iff

Proof: easy ex.

**e.g. 2** Show that $u = \bar{z}w + z\bar{w}$ is real.

A $\bar{u} =$

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**Defn** Let $z = a + bi \in \mathbb{C}$, $a$,

Its real part is

Its imaginary part is

**Formulae**

1. $Re\ z =$
2.

Proof: If \( z = a + bi \), \( a \)

**e.g. 3** Show \( u = z - \bar{z} \) is purely imaginary

i.e.

\[
u + \bar{u} =
\]

Interesting example of a field

Let \( \mathbb{F} = \mathbb{R}(t) \) be the set of real

i.e.

We’ll identify such fns if they are equal where they are both defined e.g.
We define addn & multn pointwise i.e.

\[ \text{Addn} \quad \frac{p_1(t)}{q_1(t)} + \frac{p_2(t)}{q_2(t)} := \]

\[ \text{Multn} \quad \frac{p_1(t)p_2(t)}{q_1(t)q_2(t)} := \]

Then \( \mathbb{R}(t) \) is