Lecture 23: Orthogonality & Projections

Aim Lecture The dot product can be used to study

Defn 2 vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are orthogonal

N.B. If $\mathbf{v}, \mathbf{w}$ non-zero this means the angle

We also say in this case that $\mathbf{v}, \mathbf{w}$ are normal or

Pythagoras Thm (in $\mathbb{R}^n$)

Let A, B, C be points in $\mathbb{R}^n$ with $\vec{BA}$

Proof $|AC|^2 = |\vec{BC}$

$= (\vec{BC} - \vec{BA}).$

$= |\vec{BC}|^2 +$

Unit Vectors Let $\mathbf{v} \in \mathbb{R}^n$ be non-zero.

Then $|\frac{1}{|\mathbf{v}|}\mathbf{v}| =$

i.e. $\frac{\mathbf{v}}{|\mathbf{v}|}$ unit length so called

Projections Informally

Formally: **Propn-Defn** Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,

$\mathbf{w} \neq \mathbf{0}$. There is a unique scalar $\lambda \in \mathbb{R}$ such that

$\lambda\,\mathbf{w}$ is called the

& is denoted

It's given by formula

Note 1) Scaling $\mathbf{w}$ does not change

2) $|\mathrm{proj}_{\mathbf{w}}\,\mathbf{v}\,| = |\,\mathbf{v}\,|\cos\theta$ which agrees

Proof $(\mathbf{v} - \lambda\,\mathbf{w}).\,\mathbf{w} = \mathbf{v}\,.$

$\Longrightarrow$

$\therefore\ \mathrm{proj}_{\mathbf{w}}\,\mathbf{v} =$

e.g. 1 $A(5,3), O(0,0)$.

Find coords of point B

on line $y = x$ so that

$\angle ABO$ is a right angle.

Ans $\vec{OB}$ is a scalar multiple of $\mathbf{w} =$

$\therefore$ need $\vec{OB} =$ proj

$=$

So B

N.B. B is pt on line which is

Distance from Point to Line

Q How do you find

the (shortest) distance

from pt P to line

L thru A in dirn $\mathbf{v}$?

Ans Drop perpendicular from P to pt Q

First find coords of Q by

$\vec{AQ} =$

Then compute dist of P to L as

e.g. 2 Find dist of $P(2, -1, 4)$ to line $\mathbf{x} = (1, 0, 0) + \lambda(1, -1, 1), \lambda \in \mathbb{R}$.

Ans Using notn above, A is

$\vec{AP} =$

$\vec{AQ} =$

$\vec{PQ} =$

$\therefore$ dist P to L is $|\vec{PQ}|$

Orthonormal Sets of Vectors

Defn A subset S of $\mathbb{R}^n$ is said to be orthonormal (o/n) if each $\mathbf{v} \in S$ is

e.g. 3 $S = \{\mathbf{e}_1,$

Why $|\mathbf{e}_i| =$

Mutual orthog easy to check, for example

$\mathbf{e}_1.\mathbf{e}_2 =$

e.g. 4 $\mathbf{v} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \mathbf{w} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

$\mathbf{v}, \mathbf{w}$ is

an o/n set.

Why?

The importance of o/n sets given by following generalisation of propn-defn 1.

Propn-Defn 2 Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subset \mathbb{R}^n$ be o/n.

Let $\mathbf{v} \in \mathbb{R}^n$ & $V :=$ Span

Define $\text{proj}_V \mathbf{v} :=$

1) $\mathbf{w} = \text{proj}_V \mathbf{v}$ is the unique vector in V s.t.

2) If $V = \mathbb{R}^n$

Rem a) 2) allows you to write $\mathbf{v}$ as a lin combn

b) The summands are $(\mathbf{v} \,.\, \mathbf{v}_i)\, \mathbf{v}_i =$

Proof 1) Let $\mathbf{w} = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_m \mathbf{v}_m$ be s.t. $\mathbf{v} - \mathbf{w}$ is orthog

Then $(\mathbf{v} - \mathbf{w}).\, \mathbf{v}_i = 0$ for any $i \implies$

$\mathbf{v} \,.\, \mathbf{v}_i = \mathbf{w} \,.\, \mathbf{v}_i =$

$\lambda_1 \mathbf{v}_1 \,.\, \mathbf{v}_i + \ldots + \lambda_i \mathbf{v}_i \,.\, \mathbf{v}_i +$

$=$

$\therefore \mathbf{w} = (\mathbf{v} \,.\, \mathbf{v}_1)\, \mathbf{v}_1$

Conversely, if $\mathbf{w} = \mathrm{proj}_V\, \mathbf{v}$ then above

$\therefore$ for $\alpha_1, \ldots \alpha_m \in \mathbb{R}$,

$(\mathbf{v} - \mathbf{w}). \sum \alpha_i \, \mathbf{v}_i =$

i.e. $\mathbf{v} - \mathrm{proj}_V \, \mathbf{v}$ is

Proof 2). Note 1) $\implies$ $\mathbf{v} - \mathrm{proj}_V \, \mathbf{v}$ is orthog to any vector in $V = \mathbb{R}^n$ including

$0 = (\mathbf{v} - \mathrm{proj}_V \, \mathbf{v})$.

e.g. 5 Use propn to write $\mathbf{v} = (1, 2)$ as a lin combn of a) $\mathbf{e_1}$, & $\mathbf{e_2} \in \mathbb{R}^2$,

b) $\mathbf{v} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ & $\mathbf{w} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

Ans Note in both cases sets are o/n by e.g. 3 & 4.

a) $\mathbf{v} = (\mathbf{v} \, .(1, 0))(1, 0) +$

b)